SBG STUDY

UNIT → 1

Laplace Transformation

Integral Transform

Let $K(s, t)$ be a function of two variable s, and t where s is a parameter (may be real or complex) independent of t. The function $f(s)$ defined by the integral

$$f(s) = \int_{-\infty}^{\infty} K(s,t)\, F(t)\, dt$$

It is called integral transform of the function $F(t)$ and denoted by $T\{F(t)\}$ where $K(s, t)$ is a kernal of transformation.

Laplace Transformation

Let $F(t)$ be a given function defined for all $t \geq 0$ and if the kernal $K(s, t)$ is defined as

$$K(s,t) = \begin{cases} 0 & t < 0 \\ e^{-st} & t \geq 0 \end{cases}$$

$$f(s) = \int_{-\infty}^{\infty} K(s,t)\, F(t)\, dt$$

$$= \int_{-\infty}^{0} K(s,t)\, F(t)\, dt + \int_{0}^{\infty} K(s,t)\, F(t)\, dt$$

$$f(s) = \int_{0}^{\infty} K(s,t) . F(t)\, dt$$

The $f(s)$ defined by the integral is called Laplace transformation of $f(t)$, denoted by

$$\boxed{L\{F(t)\} = f(s) = \int_{0}^{\infty} e^{-st}\, F(t)\, dt}$$

Laplace transformation of some function.

1. Find out the Laplace transformation of 1.

$$L\{1\} = \int_0^\infty e^{-st} . 1 \, dt$$

$$= \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s} \times -1 = \frac{1}{s}$$

$$L\{1\} = \frac{1}{s} = f(s)$$

2. Find out the laplace transformation e^{at}

$$L\{e^{at}\} = \int_0^\infty e^{-st} . e^{at} \, dt$$

$$= \int_0^\infty e^{-st+at} \, dt$$

$$= \int_0^\infty e^{(-s+a)t} \, dt$$

$$= \left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty$$

$$= \frac{1}{a-s}\left[0 - 1\right]$$

$$= \frac{1}{s-a}$$

3 Find out laplace transformation of sin at

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt = I$$

$$= \int e^{-st} \frac{\cos at}{a} - \frac{a}{a}\int_0^\infty e^{-st} \cos at$$

$$= \frac{1}{a}\left[e^{-st} \cos at\right] - \frac{a}{a}\left[-e^{-st} \frac{\sin at}{a} - \frac{a}{a}\int e^{-st} \sin at\right.$$

$$2I = \frac{1}{a}$$

OR

$$L\{\sin at\} = L\left\{\frac{e^{iat} - e^{-iat}}{2i}\right\}$$

$$= \frac{1}{2i}\left[L\{e^{iat} - e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[L\{e^{iat}\} - L\{e^{-iat}\}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{\cancel{s}+ia-\cancel{s}+ia}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{\cancel{2i}}\left[\frac{\cancel{2i}a}{s^2+a^2}\right]$$

$$L\{\sin at\} = \frac{a}{s^2+a^2} \quad \text{Ans}$$

4. Find out the laplace transformation of $\cos at$.

$$L\{\cos at\} = L\left\{\frac{e^{iat} + e^{-iat}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{iat}\} + L\{e^{-iat}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{ia}+s-\cancel{ia}}{(s-ia)(s+ia)}\right]$$

$$= \frac{\cancel{2}s}{\cancel{2}(s^2+a^2)}$$

$$= \frac{s}{s^2+a^2}$$

5. Find out laplace transformation of $\sinh at$.

$$L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

6. Find out laplace transformation of $\cosh at$.

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right]$$

$$= \frac{s}{s^2-a^2} \quad \text{Ans}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \text{ or } \frac{\Gamma(n+1)}{s^{n+1}}$$

7. Find out the laplace transformation of.

$$f(t) = \begin{cases} t-1 & 1<t<2 \\ 3-t & 2<t<3 \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st}.F(t)\,dt$$

$$L\{f(t)\} = \int_1^3 e^{-st}.f(t)\,dt$$

$$= \int_1^2 e^{-st}(t-1)\,dt + \int_2^3 e^{-st}(3-t)\,dt$$

$$= \int_1^2 e^{-st}\,t - \int_1^2 e^{-st} + 3\int_2^3 e^{-st} - \int_2^3 e^{-st}t$$

$$= \left[\frac{e^{-st}}{(-s)}(t-1) - \left(\frac{e^{-st}}{(-s)^2}\right)\right]_1^2 + \left[(3-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2}\right]_2^3$$

$$= \left[\frac{-(t-1)e^{-st}}{(s)} - \frac{e^{-st}}{s^2}\right]_1^2 + \left[\frac{t-3\,e^{-st}}{s} + \frac{e^{-st}}{s^2}\right]_2^3$$

$$= \frac{1}{s^2}\left[e^{-s} - 2e^{-2s} + e^{-3s}\right]$$

8. Find out the laplace transformation of function

$$f(t) = \begin{cases} t^2 & 0 < t < 2 \\ t-1 & 2 < t < 3 \\ 7 & t > 3 \end{cases}$$

$$L\{f(t)\} = \int_0^2 e^{-st} t^2\, dt + \int_2^3 e^{-st}(t-1)\, dt + \int_3^\infty e^{-st}\, 7\, dt$$

$$= \left[t^2 \frac{e^{-st}}{(-s)} - 2t\frac{e^{-st}}{(-s)^2} + 2\frac{e^{-st}}{(-s)^3}\right]_0^2 + \left[(t-1)\frac{e^{-st}}{(-s)} - \frac{e^{-st}}{(-s)^2}\right]_2^3 + 7\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \left[\frac{4e^{-2s}}{-s} - \frac{4e^{-2s}}{s^2} + \frac{2e^{-2s}}{-s^3} + \frac{2}{s^3}\right] + \left[\frac{2e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2}\right] + \frac{7}{s}\left[0 - e^{-3s}\right]$$

$$= \left[\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3} - \frac{2e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-2s}}{s^2} + \frac{7e^{-3s}}{s}\right]$$

$$= \frac{-4e^{-2s} - 2e^{-3s} + e^{-2s} + 7e^{-3s}}{s} + \frac{-4e^{-2s} - e^{-3s} + e^{-2s}}{s^2} + \frac{2 - 2e^{-2s}}{s^3}$$

$$= \frac{5e^{-3s} - 3e^{-2s}}{s} + \frac{-3e^{-2s} - e^{-3s}}{s^2} + \frac{2 - 2e^{-2s}}{s^3}$$

9. Find out laplace transformation of $L\{2\sin 2t \cos t\}$

$$L\{\sin 3t + \sin t\} = L\{\sin 3t\} + L\{\sin t\}$$
$$= \frac{3}{s^2+9} + \frac{1}{s^2+1} \text{ Ans}$$

10. Find out laplace transformation of $L\{4\sin^3 t\}$

$$L\{4\sin^3 t\} = L\{3\sin t - \sin 3t\}$$
$$= L\{3\sin t\} - L\{\sin 3t\}$$
$$= \frac{3\times 1}{s^2+1} - \frac{3}{s^2+9}$$
$$= 3\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

★ First Shifting Property

If laplace transformation,

$$L\{f(t)\} = f(s)$$
Then $L\{e^{at} \cdot F(t)\} = f(s-a)$

Ex→ $L\{e^{3t} \cdot t^2\}$

$$L\{t^2\} = \frac{2}{s^3} = f(s)$$
$$L\{e^{3t} \cdot t^2\} = \frac{2}{(s-3)^3} = f(s-3)$$

Ex→ $L\{e^{2t} \cdot \cos 3t\}$

$$L\{\cos 3t\} = \frac{s}{s^2+9} = f(s)$$
$$L\{e^{2t} \cdot \cos 3t\} = \frac{s-2}{(s-2)^2+9} = f(s-2)$$

3. $L\{e^{-t}\sin 4t\}$

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{e^{-t}\sin 4t\} = \frac{4}{(s+1)^2+16}$$

4. $L\{e^{-t}\cdot\cos^2 t\}$

$$= \frac{1}{2}L\{e^{-t}\cos 2t\} + \frac{1}{2}L\{e^{-t}\}$$

$$= \frac{1}{2}\,\frac{s+1}{(s+1)^2+4} + \frac{1}{2}\left(\frac{1}{s+1}\right)$$

5. $L\{t^2\cdot e^{-4t}\}$

$$L\{e^{-4t}\,t^2\} = L\{(t)^2\} = \frac{2}{s^3} = f(s)$$

$$= \frac{2}{(s+4)^3} = f(s+4)$$

6. $L\{e^{-t}\cos 2t\cos t\}$

$$= \frac{1}{2}L\left\{e^{-t}\left(\cos 3t + \cos t\right)\right\}$$

$$= \frac{1}{2}L\{e^{-t}\cos 3t\} + \frac{1}{2}L\{e^{-t}\cos t\}$$

$$= \frac{1}{2}\left(\frac{s+1}{(s+1)^2+9}\right) + \frac{1}{2}\left(\frac{s+1}{(s+1)^2+1}\right)$$

Second shifting properties

If $L\{F(t)\} = f(s)$

$$G(t) = \begin{cases} F(t-a) & t>a \\ 0 & t<a \end{cases}$$

Then $L\{G(t)\} = e^{-as} \cdot f(s)$

1. Find out the laplace transformation of $f(t)$:

$$f(t) = \begin{cases} \sin(t-\pi/3); & t>\pi/3 \\ 0 \quad ; & t<\pi/3 \end{cases}$$

$$= e^{-\pi/3 s} \; L\{\sin t\}$$

$$= e^{-s\pi/3} \cdot \frac{1}{s^2+1}$$

Multiply by t^n

If $F(t)$ is a function of class A and if t

$$L\{F(t)\} = f(s)$$

$$L\{t^n . F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

1. Find out the laplace transformation of $f(t)$:-

$$L\{t . e^{-t} \sin 2t\}$$

$$L\{\sin 2t\} = \frac{2}{s^2+4} = f(s)$$

$$L\{e^{-t} \sin 2t = \frac{2}{(s+1)^2+4}$$

$$L\{t e^{-t} \sin 2t = (-1)^1 \, 2\frac{d}{ds} \frac{1}{(s+1)^2+4}$$

$$= -2 \left(\frac{-2(s+1)}{((s+1)^2+4)^2} \right)$$

$$= \frac{4(s+1)}{((s+1)^2+4)^2}$$

3. $L\{t^2 e^t \sin^4 t\}$

$$L\{\sin 4t\} = \frac{4}{s^2+16} = f(s)$$

$$L\{e^t \sin 4t\} = \frac{4}{(s-1)^2+16} = f(s-1)$$

$$L\{t^2 e^t \sin 4t\} = 4\frac{d^2}{ds^2}\frac{1}{(s-1)^2+16}$$

$$= 4\frac{d}{ds}\frac{-2(s-1)}{((s-1)^2+16)^2}$$

$$= -8\frac{d}{ds}\frac{(s-1)}{((s-1)^2+16)^2}$$

$$= -8 \times \frac{[(s-1)^2+16]^2 - (s-1)\,2((s-1)^2+16) \times 2(s-1)}{((s-1)^2+16)^4}$$

$$= -8\frac{[(s+1)^2+16]\{1 - 4(s-1)^2]}{((s-1)^2+16)^4}$$

$$= \frac{-8[1 - 4(s-1)^2]}{((s-1)^2+16)^3}$$

$$= -8(1 - 4s^2.$$

$$= -8\frac{[(s-1)^2+16][s^2+1-2s+16-4s^2-4+8s]}{((s-1)^2+16)^4}$$

$$= \frac{-8[13-3s^2+6s]}{(s^2-2s+17)^3}$$

4. $L\{t e^{-t} \cosh t\}$

$$L\{\cosh t\} = \frac{s}{s^2-1}$$

$$L\{t \cosh t\} = (-1)^1 \frac{d}{ds} \frac{s}{(s^2-1)}$$

$$= \left[\frac{(s^2-1) - s(2s)}{(s^2-1)^2}\right]$$

$$= -\left[\frac{s^2-1-2s^2}{(s^2-1)^2}\right]$$

$$= \frac{s^2+1}{(s^2-1)^2}$$

$$= \frac{(s+1)^2+1}{((s+1)^2-1)^2}$$

$$= \frac{s^2+2+2s}{(s^2+2s\ )^2}$$

<u>Division by t^n</u>

$L\{F(t)\} = f(s)$

Then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\, ds$

1 Find out Laplace transformation of:

$$L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$$

$$= L\left\{\frac{e^{-at}}{t}\right\} - L\left\{\frac{e^{-bt}}{t}\right\}$$

$$= L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \frac{1}{s+a} - \frac{1}{s+b} = f(s)$$

$$\int_s^\infty f(s) = \int_s^\infty \frac{1}{s+a} - \frac{1}{s+b}\, ds$$

$$= \left[\log(s+a) - \log(s+b) \right]_s^\infty$$

$$= \left[\log \frac{s+a}{s+b} \right]_s^\infty$$

$$= \left[\log \frac{(1 + a/s)}{(1 + b/s)} \right]_s^\infty$$

$$= \log 1 - \log \frac{(s+a)}{(s+b)}$$

$$= -\log \frac{(s+a)}{(s+b)}$$

$$= \log \left(\frac{s+b}{s+a} \right)$$

2 Find out laplace transformation of:

$$L \left\{ \frac{\cos at - \cos bt}{t} \right\}$$

$$= L\{\cos at - \cos bt\}$$

$$= L\{\cos at\} - L\{\cos bt\}$$

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = f(s)$$

$$L\{f(s)\} = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \right] ds$$

let $s^2+a^2 = x$

$$2s\,ds = dx$$

$$s\,ds = \frac{1}{2} dx$$

$$\frac{1}{2} \left[\log(x) - \log(y) \right]_{s^2+a^2}^\infty$$

$$\frac{1}{2} \left[\log \frac{x}{y} \right]$$

$$\frac{1}{2} \left[\log \frac{s^2+a^2}{s^2+b^2} \right]_s^\infty$$

$$\frac{1}{2} \log \left[\frac{1 + \frac{a^2}{s^2}}{1 + \frac{b^2}{s^2}} \right]_s^\infty$$

$$\int \log x\,dx = x\log x - x + C = x(\log x - 1) + C$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

3. $$L\left\{\frac{1-\cos t}{t^2}\right\}$$

$$= L\left\{\frac{1}{t^2}\right\} - L\left\{\frac{\cos t}{t^2}\right\}$$

$$= \cancel{L\{e^{t}\}} = L\{1-\cos t\}$$

$$= \left[\frac{1}{s} - \frac{s}{s^2+1}\right] = f(s)$$

$$\int_s^\infty f(s) = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty$$

$$= \left[\log \frac{1}{\sqrt{1+\frac{1}{s^2}}}\right]_s^\infty$$

$$= \log 1 - \log \frac{s}{\sqrt{s^2+1}}$$

$$f(s_2) = \log \frac{\sqrt{s^2+1}}{s}$$

$$\int_s^\infty f(s_2) = \int_s^\infty \log \frac{\sqrt{s^2+1}}{s} \cdot 1$$

$$= \frac{1}{2}\log \frac{s^2+1}{s^2} \cdot s - \frac{1}{2}\int_0^\infty \frac{2s \times s^2 - (s^2+1)2s}{s^4} \cdot s = \left[\frac{\sqrt{s^2+1}}{s}\left(\log \frac{\sqrt{s^2+1}}{s} - 1\right)\right]_s^\infty$$

$$\left[\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right) \cdot s\right]_0^\infty - \frac{1}{2}\int_0^\infty \frac{2s}{s^4} \times \frac{s^4}{(s^2+1)} \cdot s = \left[\sqrt{1+\frac{1}{s^2}}\left(\log\sqrt{1+\frac{1}{s^2}} - 1\right)\right]_s^\infty$$

$$\frac{s}{2} \log \frac{s^2}{s^2+1} + \cot^{-1} s = -1 - \sqrt{\frac{s^2+1}{s}}\left[\log \frac{\sqrt{s^2+1}}{s} - 1\right] \text{ Ans}$$

2. $L\left\{\frac{e^{at}-\cos bt}{t}\right\}$

$L\{e^{at}-\cos bt\} = \frac{1}{s-a} - \frac{s}{s^2+b^2} = f(s)$

$\int_s^{\infty} f(s) = \int_s^{\infty}\left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right) ds$

$= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^{\infty}$

$= \left[\log \frac{s-a}{\sqrt{s^2+b^2}}\right]_s^{\infty}$

$= \left[\log \frac{\cancel{s}(1-a/s)}{\cancel{s}\sqrt{1+\frac{b^2}{s^2}}}\right]_s^{\infty}$

$= \log 1 - \log \frac{s-a}{\sqrt{s^2+b^2}}$

$= \log \frac{\sqrt{s^2+b^2}}{s-a}$ Ans

3. $L\left\{\frac{e^{2t}\cdot\cos 4t}{t}\right\}$

$L\{e^{2t}\cdot\cos 4t\} = \frac{s}{s^2+16} = f(s)$

$= L\{f(s) = \frac{s-2}{(s-2)^2+16}$

$= \int_s^{\infty} \frac{s-2}{(s-2)^2+16}\, ds$

$= \frac{1}{2}\left[\log\left((s-2)^2+16\right)\right]_s^{\infty}$

$= \frac{1}{2}\Big[$

- **Laplace transform of derivative of $f(t)$**

If $L\{F(t)\} = f(s)$
Then $L\{F'(t)\} = sL\{F(t)\} - F(0)$

Laplace transform of Integral

If $f(t)$ is piece wise continuous function and satisfy $|F(t)| = Me^{\alpha t}$; $t \geq 0$

$$L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{s} f(s) = \frac{1}{s} L\{F(t)\}$$

1. $\displaystyle\int_0^t \frac{e^{at} - e^{bt}}{t}\,dt$

$$= \frac{1}{s} f(s) = \frac{\log\left(\frac{s+b}{s+a}\right)}{(s)}$$

2. $\displaystyle F(t) = \int_0^t \frac{\sin t}{t}\,dt$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$\int_s^\infty \frac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = f_1(s)$$

$$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{f_1(s)}{s} = \frac{\cot^{-1} s}{s}$$

★ 3. $\displaystyle\int_0^{t/2} \frac{1-e^{-2x}}{x}\,dx$

let $t = 2x$

$dt = 2dx$

$$= \int_0^t \frac{1-e^{-t}}{t/2} \times \frac{1}{2}\,dt$$

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★ 3. $L\left\{ \int_0^{t/2} \frac{1-e^{-2x}}{x}\, dx \right\}$

$$2x = u$$

$$2\,dx = du$$

$$L\left\{ \int_0^{t} \frac{1-e^{-u}}{\frac{u}{2}} \times \frac{1}{2}\, du \right\}$$

$$L\left\{ 1-e^{-u} \right\} = \frac{1}{s} - \frac{1}{s+1} = f(s)$$

$$L\left\{ \frac{1-e^{-u}}{u} \right\} = \int_0^{\infty} f(s)\, ds = \left[\log \frac{s}{s+1} \right]_0^{\infty}$$

$$= -\log \frac{s}{s+1}$$

$$= \frac{1}{s} \log\left(\frac{s+1}{s}\right)$$

4. Show that:-

$$L\left\{ \frac{\cos\sqrt{t}}{\sqrt{t}} \right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4s}}$$

$$\cancel{\text{Let } \sqrt{t} = t}$$

$$L\left\{ \frac{\cos\sqrt{t}}{\sqrt{t}} \right\}$$

$$\cancel{L\left\{ \cos^2\sqrt{t} \right\}}$$

$$\cancel{L\left\{ \frac{1+\cos 2\sqrt{t}}{2} \right\} = \frac{1}{2} L\{1\} + \frac{1}{2} L\{\cos 2\sqrt{t}\}}$$

$$\cancel{= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+} \right.}$$

$$= \int_0^{\infty} e^{-st} \left(\frac{\cos\sqrt{t}}{\sqrt{t}} \right) dt$$

$$= \int_0^{\infty} e^{-st}\, \frac{e^{i\sqrt{t}} + e^{-i\sqrt{t}}}{2\sqrt{t}}\, dt$$

Let $\sqrt{t} = v$

$$\frac{1}{2\sqrt{t}}\, dt = dv$$

$$= \int_0^\infty e^{-sv^2}\left(e^{iv}+e^{-iv}\right)dv$$

$$= e^{-\frac{1}{4s}}\int_0^\infty e^{-s\left(v-\frac{i}{2s}\right)^2} + e^{-s\left(v+\frac{i}{2s}\right)^2}\, dv$$

Let $x = \sqrt{s}\left(v-\frac{i}{2s}\right)$ and $y = \sqrt{s}\left(v+\frac{i}{2s}\right)$

$$= \frac{e^{-1/4s}}{\sqrt{s}}\int_0^\infty e^{-x^2}dx + \int_0^\infty e^{-y^2}dy$$

$$= \frac{e^{-1/4s}}{\sqrt{s}}\left(\frac{\sqrt{\pi}}{2}+\frac{\sqrt{\pi}}{2}\right) = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$

5. Evaluate :- $\int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t}\,dt$

Comparing with $\int_0^\infty e^{-st} f(t)\,dt = f(s)$

$$= \int_0^\infty e^{-t}\frac{\sin\sqrt{3}t}{t} = f(1)$$

$$= \int_s^\infty \frac{\sqrt{3}}{s^2+3}\,ds$$

$$= \frac{\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}}\left[\tan^{-1}\frac{s}{\sqrt{3}}\right]_s^\infty$$

$$\cot^{-1}\frac{s}{\sqrt{3}} = f_1(s)$$

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = f_1(1)$$

$$= \cot^{-1}\frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{3}\ \text{Ans}$$

6. Evaluate :- $\int_0^\infty \frac{e^{-2t}-e^{-4t}}{t}\,dt$

$$= \int_0^\infty e^{-2t}\frac{(1-e^{-2t})}{t}\,dt$$

$$= \int_s^\infty L\{(1-e^{-2t})\}\,ds$$

$$= \int_s^\infty\left(\frac{1}{s}-\frac{1}{s+2}\right)ds$$

$$= \log\left(\frac{s+2}{s}\right)$$

$$\int_0^\infty e^{-2t} f(t) = \log\left(\frac{2+2}{2}\right) = \log 2$$

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7. Prove that $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt$

$s = 1.$

$L\{\sin^2 t\} = \frac{1}{2} L\{1 - \cos 2t\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+4}$

$= \frac{1}{2}\left[\log \frac{s}{\sqrt{s^2+4}}\right]_s^\infty$

$= \frac{1}{4}\left[\log \frac{s^2}{s^2+4}\right]_s^\infty$

$= \frac{1}{4} \log\left[\frac{s^2+4}{s^2}\right] = f(1)$

$\int_0^\infty \left\{e^{-t} \frac{\sin^2 t}{t}\right\} dt = \frac{1}{4} \log 5$

8. $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$

$\int_0^\infty e^{0t} \; F(t)\, dt$

$s = 0;$

$L\left\{\frac{\cos 6t - \cos 4t}{t}\right\}$

$\int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds$

$\frac{1}{2}\left[\log (s^2+36) - \log s^2+16\right]_s^\infty$

$\frac{1}{2} \log \left[\frac{s^2+36}{s^2+16}\right]_s^\infty$

$\frac{1}{2} \log\left[\frac{s^2+16}{s^2+36}\right]$

$\frac{1}{2} \log\left(\frac{4}{9}\right) = \log \frac{2}{3}$

Unit Step function (Heaviside unit step function)

The unit step function $[U(t-a)]$ is defined as

$$U(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t \geq a \end{cases}$$

Or in particular case,

$$U(t) = \begin{cases} 0 & ; t < 0 \\ 1 & ; t \geq 0 \end{cases}$$

Product $\Rightarrow$ $F(t)\,U(t-a) = \begin{cases} 0 & ; t < a \\ F(t) & ; t \geq a \end{cases}$

Q. Find out the laplace transformation of unit step funct.

$$L\{F(t)\} = \int_0^\infty e^{-st}.F(t)\,dt$$

$$L\{U(t-a)\} = \int_0^\infty e^{-st}\,U(t-a)\,dt$$

$$= \int_0^a e^{-st}\,U(t-a)\,dt + \int_a^\infty e^{-st}\,U(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{e^{-as}}{s} \quad \text{Ans}$$

Second shifting property in unit step function

$L\{F(t)\} = f(s)$, then $L\{F(t-a)\,U(t-a)\} = e^{-as}\,L\{F(t)\}$

or

$= e^{-as} f(s)$

Q. $L\{(t-2)^2\,U(t-2)\} =$

$$= e^{-2s}\,L\{t^2\}$$

$$= e^{-2s} \cdot \frac{2}{s^3}$$

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Q. $L\{e^{(t-2)} U(t-2)\}$

$= e^{-2s} L\{e^{t}\}$

$= e^{-2s}\left(\frac{1}{s-1}\right)$

Q. Express the following function in term of unit step function and find out laplace transformation

$$f(x) = \begin{cases} 0 & 0<t<1 \\ (t-1) & 1<t<2 \\ 1 & 2<t \end{cases}$$

$F(t) = 0\{U(t-0)-U(t-1)\} + t-1\{U(t-1)-U(t-2)\} + 1\{U(t-2)\}$

$f(x) = t-1\{U(t-1)\} - (t-1)(U(t-2) + U(t-2)$

$= t-1\{U(t-1)\} - U(t-2)\{t-1-1\}$

$= t-1\{U(t-1)\} - (t-2)U(t-2)$

$L\{F(t)\} = L\{(t-1)U(t-1)\} - L\{(t-2)(U(t-2)\}$

$= e^{-s} L\{t\} - e^{-2s} L\{t\}$

$= \frac{e^{-s} - e^{-2s}}{s^2}$ Ans

Q. $f(x) = \begin{cases} (t-1) & 1<t<2 \\ (3-t) & 2<t<3 \end{cases}$

$F(t) = t-1\{U(t-1) - U(t-2)\} + (3-t)(U(t-2) - U(t-3)$

$= (t-1)U(t-1) - (t-1)U(t-2) + (3-t)(U(t-2) + (t-3)U(t-3)$

$= (t-1)U(t-1) - U(t-2)\{t-1-3+t\} + (t-3)U(t-3)$

$= (t-1)U(t-1) - U(t-2)\,2(t-2) + (t-3)U(t-3)$

$L\{F(t)\} = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$

Periodic Function

A function $f(t)$ is said to be periodic if

$$F(t) = F(t+T) = F(t+2T) = \dots\dots$$

where T is positive constant and is called period of $F(t)$.

Find out the laplace transformation of $f(t)$ where $f(t)$ is periodic function.

Let $F(t)$ is a periodic function

$$L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt$$

Q Draw graph of following periodic function and find out the laplace transformation:-

$$F(t) = \begin{cases} t & \text{for } 0<t<a \\ 2a-t & \text{for } a<t<2a \end{cases}$$

$F(t)$ is a periodic function.

$$L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} F(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st}(2a-t)\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[t\left(\frac{e^{-st}}{-s}\right) - \frac{e^{-st}}{(-s)^2} \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) + \frac{e^{-st}}{(-s)^2} \right]_a^{2a}$$

$$= \frac{1}{1-e^{-2as}} \left[\cancel{\frac{a e^{-as}}{-s}} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \cancel{\frac{a e^{-as}}{-s}} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{1}{s^2(1-e^{-2as})} \left[1 + e^{-2as} - 2e^{-as} \right]$$

Q. Find L {square wave function} of period defined as,

$$F(t) = \begin{cases} 1 & 0 < t < a/2 \\ -1 & a/2 < t < a \end{cases}$$

$$L\{F(t)\} = \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^{a} e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[-\frac{e^{-as/2}}{s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-as/2}}{s}\right]$$

$$= \frac{1}{s(1-e^{-as})}\left[1 + e^{-as} - 2e^{as/2}\right]$$

Inverse Laplace Transform

$L\{F(t)\} = f(s)$ then $F(t)$ is known as inverse laplace transform and denoted by

$$F(t) = L^{-1}\{f(s)\}$$

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$$

Ex→ $L^{-1}\left\{\frac{1}{s^2+36}\right\} = \frac{1}{6}\sin 6t$

$L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$

$L^{-1}\left\{\frac{2s-1}{s^2+4}\right\} = 2\cos 2t - \frac{1}{2}\sin 2t$

Q. $L^{-1}\left\{\frac{4s+15}{16s^2+25}\right\}$

$= \frac{4s}{16s^2+25} + \frac{15}{16}\left\{\frac{1}{s^2+\left(\frac{5}{4}\right)^2}\right\}$

$= \frac{1}{4}\cos\frac{5t}{4} + \frac{\cancel{15}^{3}}{4\cancel{16}} \times \frac{\cancel{4}}{\cancel{5}}\sin\frac{5}{4}t$

$\frac{1}{4}\cos\frac{5t}{4} + \frac{3}{4}\sin\frac{5}{4}t$

1st shifting property

$L^{-1}\{f(s)\} = F(t)$

$L^{-1}\{f(s-a)\} = e^{at}\, L^{-1}\{f(s)\}$ or $e^{at}\, F(t)$

Ex→ $L^{-1}\left\{\frac{1}{(s-2)^5}\right\} = e^{2t}\, L\left\{\frac{1}{s^5}\right\} = e^{2t}\frac{t^4}{4!}$

$L^{-1}\left\{\frac{s-4}{(s-4)^2+25}\right\} = e^{4t}\, L\left\{\frac{s}{s^2+25}\right\} = e^{4t}\cos 5t$

Q. $L^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$

$L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$

Q. $L^{-1}\left\{\frac{1}{s^2+s+1}\right\} = L^{-1}\left\{\frac{1}{s^2+\left(\frac{1}{\sqrt{2}}\right)^2+s-}\right\}$ $\frac{e^{-1/2}}{\sqrt{3}/2}\sin\frac{\sqrt{3}t}{2}$

Q. $L^{-1}\left\{\frac{1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$

$= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$

$= 1 - e^{-t}$ Ans

Q. $L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$

$5s+3 = A(s^2+2s+5) + Bs(s-1) + C(s-1)$

$5s+3 = A(s^2+2s+5) + B(s^2-s) + C(s-1)$

$5s+3 = (A+B)s^2 + (2A-B+C)s + 5A-C$

$A+B=0 \qquad 2A-B+C=5 \qquad 5A-C=3$

$B=-1 \qquad 2A+A+5A-3=5 \qquad C=2$

$8A=8$

$A=1$

$L^{-1} = \left\{\frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}\right\}$

$= e^{t} + \frac{(-s+2)}{(s+1)^2+2^2}$

$= e^{t} - \frac{(s+1)-3}{(s+1)^2+2^2}$

$= e^{t} - e^{-t}\cos 2t + e^{-t}\frac{3}{2}\sin 2t$

• $L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\}$

$L^{-1}\left\{\frac{s}{(s-2)^2+4}\right\} = L^{-1}\left\{\frac{(s-2)}{(s-2)^2+2^2} + \frac{2}{(s-2)^2+2^2}\right\}$

$= e^{2t}L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 2L^{-1}\left\{\frac{1}{s^2+2^2}\right\}$

$= e^{2t}\cos 2t + \frac{2}{2}e^{2t}\sin 2t$

$= e^{2t}\cos 2t + e^{2t}\sin 2t$

Inverse laplace transform of derivatives

1. $L^{-1}\{F(s)\} = F(t)$
2. $L^{-1}\left\{\frac{d^n (s)}{ds^n}\right\} = (-1)^n t^n \cdot F(t)$

Multiplication power of s

1. $L^{-1}\{F(s)\} = F(t)$

$L^{-1}\{s^n F(s)\} = \frac{d^n}{dt^n}(F(t))$

Division by power of s

1. $L^{-1}(F(s)) = F(t)$

$L^{-1}\left\{\frac{1}{s} F(s)\right\} = \int_0^t F(x)\,dx = F_1(t)$

$L^{-1}\left\{\frac{1}{s^2} F(s)\right\} = \int_0^t F_1(x)\,dx = F_2(t)$

$L^{-1}\left\{\frac{1}{s^3} F(s)\right\} = \int_0^t F_2(x)\,dx = F_3(t)$

Q. $L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = L^{-1}\left\{s \cdot \frac{s}{s^2+1}\right\}$

$\because L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cancel{\sin} \cos t = F_1(t)$

$\therefore L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = \frac{d\cos t}{dt} = \cancel{\cos t}\ -\sin t.$

$L^{-1}\{F^n(s)\} = (-1)^n t^n F(t)$

Q. $L^{-1}\left\{\log\frac{(s+a)}{(s+b)}\right\}$

$F(s) = \log\frac{(s+a)}{(s+b)} = \log(s+a) - \log(s+b)$

~~F(s) = log b~~ differentiate w.r.t s

$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

Taking L^{-1} both sides,

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\}$$

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{1}{s+b}\right\}$$

$$(-1)^1 t^1 F(t) = e^{-at} - e^{-bt}$$

$$F(t) = \frac{-(e^{-at} - e^{-bt})}{t}$$

3. $L^{-1}\left\{\cot^{-1}\left(\frac{s+3}{2}\right)\right\}$

$$F(s) = \cot^{-1}\left(\frac{s+3}{2}\right)$$

differentiate w.r.t s.

$$F'(s) = -\frac{1}{1+\left(\frac{s+3}{2}\right)^2} \times \frac{1}{2}$$

$$F'(s) = \frac{-1/\cancel{2}}{\frac{4+(s+3)^2}{4}}$$

$$F'(s) = \frac{-2}{4+(s+3)^2}$$

Taking L^{-1} both sides,

$$L^{-1} F'(s) = L^{-1}\left(\frac{-2}{4+(s+3)^2}\right)$$

$$= -2e^{-3t} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{-\cancel{2}e^{-3t}\sin 2t}{\cancel{2}}$$

$$(-1) t^1 F(t) = -e^{-3t} \sin 2t$$

$$F(t) = \frac{e^{-3t}\sin 2t}{t}$$

Convolution Theorem

$$L^{-1}\{f(s)\} = F(t)$$

$$L^{-1}\{g(s)\} = G(t)$$

$$L^{-1}\{f(s)\cdot g(s)\} = \int_0^t F(x)\, G(t-x)\, dx$$

Q. Using convolution theorem find $L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\}$

$$L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\} = \int_0^t \cos 2x \cos(t-x)\, dx$$

$$= \frac{1}{2}\int_0^t 2\cos 2x \cos(t-x)\, dx$$

$$= \frac{1}{2}\int_0^t \cos\left(\frac{2x+t-x}{2}\right) + \cos\left(\frac{2x-t+x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^t \cos\left(\frac{x+t}{\cancel{2}}\right) + \cos\left(\frac{3x-t}{\cancel{2}}\right) dx$$

$$= \frac{1}{2}\left[\cancel{2}\sin\left(\frac{x+t}{\cancel{2}}\right) + \frac{\cancel{2}}{3}\sin\left(\frac{3x-t}{\cancel{2}}\right)\right]_0^t$$

$$= \left[\sin\left(\frac{x+t}{\cancel{2}}\right) + \frac{1}{3}\sin\left(\frac{3x-t}{\cancel{2}}\right)\right]_0^t$$

$$= \frac{1}{2}\left[\sin 2t + \frac{1}{3}\sin t - \sin\frac{t}{\cancel{2}} - \frac{1}{3}\sin\left(\frac{-t}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{4}{3}\sin 2t - \frac{2}{3}\sin\frac{t}{2}\right]$$

$$= \frac{2}{3}\sin 2t - \frac{1}{3}\sin\frac{t}{\cancel{2}}$$

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Q2. $L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

$$L^{-1}\left\{\frac{s}{s^2+4}\cdot\frac{1}{s^2+4}\right\} = \frac{1}{2}\int_0^t \cos 2x \,\sin(2t-2x)\,dx$$

$$= \frac{1}{4}\int_0^t 2\,\cancel{\sin(t\ldots)}\,\cos 2x\,\sin(t-2x)\,dx$$

$$\cancel{\sin(2x-t) - \sin}$$

$$= \frac{1}{4}\int_0^t \sin 2t - \sin(4x-2t)\,dx$$

$$= \frac{1}{4}\sin 2t\,(t) - \frac{1}{4}\left[\frac{\cos(4x-2t)}{4}\right]_0^t$$

$$= \frac{1}{4}\sin 2t\,(t) - \frac{1}{4}\left[\cancel{\frac{\cos 2t}{4}} - \cancel{\cos 2t}\right]$$

$$= \frac{t}{4}\sin 2t$$

Q3. Using convolution theorem,
Find $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

$$L^{-1}\left\{\frac{1}{s^3}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \frac{x^2}{2}\cdot\sin(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t x^2 \sin(t-x)\,dx$$

$$\cancel{\frac{1}{2}\left[x^2\cos(t-x) + 2x\sin(t-x) + 2\cos(t-x)\right]}$$

$$= \frac{1}{2}\left[t^2 - 0 + 2 - 2\cos t\right]$$

$$= \frac{1}{2}\left[-2\cos t + t^2 + 2\right] \quad \text{Ans}$$

$$= \frac{t^2}{2} - \cos t + 1$$

$$L^{-1}\left\{\frac{1}{s^3}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \frac{x^2}{2}\cdot \sin(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t x^2 \sin(t-x)\,dx$$

$$= \frac{1}{2}\left[x^2\left(\frac{\cos(t-x)}{-1}\right) - 2x\left\{-\frac{\sin(t-x)}{(-1)^2}\right\} + \frac{2\cos(t-x)}{(-1)^3}\right]$$

$$= \frac{1}{2}\left[x^2 \cos(t-x) + 2x\sin(t-x) - 2\cos(t-x)\right]_0^t$$

$$= \frac{1}{2}\left[t^2 - 2 + 2\cos t\right]$$

$$= \frac{t^2}{2} + \cos t - 1 \quad \text{Ans}$$

Q4. Show that

$$1 \times 1 \times 1 \times \ldots\ldots 1 \text{ (n times)} = \frac{t^{n-1}}{\lfloor n-1} \quad \text{where } n = 1, 2, 3 \ldots$$

When $n = 2$,

$$1 \times 1 \Rightarrow \int_0^t 1.1\,dt = t$$

When $n = 3$,

$$1 \times \underbrace{1 \times 1} = 1 \times t = \int_0^t 1.t = \frac{t^2}{2!}$$

When $= 4$,

$$1 \times \underbrace{1 \times 1 \times 1} = 1 \times \frac{t^2}{2} = \int_0^t 1.\frac{t^2}{2} = \frac{t^3}{3!}$$

When $n = 5$

$$1 \times \underbrace{1 \times 1 \times 1 \times 1} = 1 \times \frac{t^3}{3!} = \frac{1}{3!}\int_0^t t^3 = \frac{1}{3!} \times \frac{t^4}{4} = \frac{t^4}{4!} = \frac{t^{n-1}}{(n-1)!}$$

When $n = n$,

$$1 \times 1 \times 1 \times 1 \ldots n \text{ (times)} = \frac{t^{n-1}}{(n-1)!} \quad \text{proved}$$

Application to differential equation

Methods for solution

First of all take laplace transform on both side of given problem and then use the formula of Laplace transform of derivative.

$$L\{Y'(t)\} = s\,L\{y(t)\} - Y(0)$$

$$L\{Y''(t)\} = s^2 L\{Y(t)\} - s\,Y(0) - Y'(0)$$

$$L\{Y'''(t)\} = s^3 L\{Y(t)\} - s^2 Y(0) - s\,Y'(0) - Y''(0)$$

Q. Using Laplace transform, find the value of initial value problem.

$$y'' \cancel{+} 4y' + 4y = 64 \sin 2t$$

where $y(0) = 0,\ y'(0) = 1$

Taking laplace both side

$$L\{y'' - 4y' + 4y\} = 64 L\{\sin 2t\}$$

$$L\{y''\} - 4L\{y'\} + 4L\{y\} = 64 L\{\sin 2t\}$$

$$\{s^2 \bar{y} - sy(0) - y'(0)\} - \{4s\bar{y} - 4y(0)\} + 4\bar{y} = 64 \cdot \frac{2}{s^2+2^2}$$

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\bar{y} = 64 \cdot \frac{2}{s^2+2^2} + 1$$

$$\bar{y}(s^2 - 4s + 4) = 64 \cdot \frac{2}{s^2+4} + 1$$

$$\bar{y}\,(s-2)^2 = \frac{64 \cdot 2}{s^2+4} + 1$$

$$\bar{y} = \frac{1}{(s-2)^2} + \frac{128}{(s^2+4)(s-2)^2}$$

$$y = L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 128 L^{-1}\left\{\frac{1}{(s^2+4)(s-2)^2}\right\}$$

$$\frac{128}{(s-2)^2(s^2+4)} = \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{Cs+D}{(s^2+4)}$$

$$= \frac{-8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s+0}{s^2+4}$$

$$\bar{y} = L^{-1}\left\{\frac{1}{(s-2)^2} - \frac{8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s}{s^2+4}\right\}$$

$$y = L^{-1}\left\{\frac{17}{(s-2)^2} - \frac{8}{s-2} + \frac{8s}{s^2+4}\right\}$$

$$y = 17\, L^{-1}\left\{\frac{1}{(s-2)^2}\right\} - 8\, L^{-1}\left\{\frac{1}{s-2}\right\} + 8\, L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= 17 e^{2t} t - 8 e^{2t} + 8 \cos 2t$$

Q Solve the differential equation:-

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

$$L\{y'''\} + 2L\{y''\} - L\{y'\} - 2\bar{y} = 0$$

$$s^3\bar{Y} - s^2 - s\,2 - 2 + 2s^2\bar{Y} - 2s - 2 - s\bar{Y} + 1 - 2\bar{Y} = 0$$

$$\bar{Y}(s^3 + 2s^2 - s - 2) - (s^2 + 4s - 3) = 0$$

$$\bar{Y} = \frac{s^2 + 4s + 5}{s^3 + 2s^2 - s - 2}$$

$$\bar{Y} = \frac{s^2 + 4s + 5}{(s-1)(s^2 - 2s + 2)}$$

$$\bar{Y} = \frac{s^2 + 4s + 5}{(s-1)(s+1)(s+2)}$$

$A = \frac{5}{3}$
$B = -1$
$C = \frac{1}{3}$

$$y = L^{-1}\left\{\frac{s^2 + 4s + 5}{(s-1)(s+1)(s+2)}\right\}$$

$$y = \frac{5}{3} L^{-1}\left\{\frac{1}{s-1}\right\} - 1\, L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3} L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{5}{3} e^{t} - e^{-t} + \frac{1}{3} e^{-2t} \quad \text{Ans}$$

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Q. Solve :-

$y'' + y = t\cos 2t$

Given that $y(0) = 0$, $y'(0) = 0$

$L\{y'' + y\} = L\{t\cos 2t\}$

$= (-1) \frac{d}{ds} \frac{\sin 2t}{s^2+4}$

Let $s^2+4 = d$, $2s\,ds = dd$, $s\,ds = \frac{dd}{2}$

$L\{y''\} + L\{y\} = \cancel{2\sin 2t}$ $-\frac{1}{2}\log(s^2+4) = \log\frac{1}{\sqrt{s^2+4}}$

$s^2\bar{Y} - sY(0) - sY'(0) + \bar{Y} = 2\sin 2t$

$\bar{Y}(s^2+1) = 2\sin 2t$

$\bar{Y} = \frac{2\sin 2t}{s^2+1}$

$L(Y) = 2\frac{\sin 2t}{s^2+1}$

$Y = 2L^{-1}\left\{\frac{\sin 2t}{s^2+1}\right\}$

$L\{y'' + y\} = L\{t\cos 2t\}$

$s^2\bar{Y} - sY(0) - Y'(0) + \bar{Y} = \frac{d}{ds}\left(\frac{s}{s^2+4}\right)$

$\bar{Y}(s^2+1) = \frac{s^2+4 - s(2s)}{(s^2+4)^2}$

$\bar{Y}(s^2+1) = \frac{-s^2+4}{(s^2+4)^2}$

$\bar{Y} = \frac{-s^2+4}{(s^2+4)^2} \times \frac{1}{s^2+1}$

$y = L^{-1}\left\{\frac{-s^2+4}{(s^2+4)^2(s^2+1)}\right\}$

SBG STUDY