

10/10/18

## Unit - 5 (Partial derivatives)

Let  $z = f(x, y)$  be a given fn of two independent variables  $x, y$ .  
then if we differentiate  $z$  w.r.t  $x$  treating  $y$  as constant then the differentiation is called partial differentiation of  $z$  w.r.t  $x$  and it is denoted by

$$\frac{\partial z}{\partial x}$$

$$\therefore \frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q.$$

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)$$

$$\frac{\partial^2 z}{\partial y^2} = t$$

Q. if  $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

find the value of

$$\frac{x \partial u}{\partial x} + \frac{y \partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{-y}{x^2}$$



$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$x \frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \left( -\frac{x}{y^2} \right) + \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y \sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$y \frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \quad (2)$$

$$(1) + (2)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

Q.  $u = (1 - 2xy + y^2)^{-1/2}$   
P.T.

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = y^2 u^3$$

$$x \frac{\partial u}{\partial x} = -\frac{1}{2} (1 - 2xy + y^2)^{-3/2} \times (-2y)$$

$$\frac{\partial u}{\partial x} = y u^3$$



$$a^x = e^{x \log a}$$

$$x \frac{\partial u}{\partial x} = x y^4^3$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2} y^3 (-2x + 2y)$$

$$\frac{\partial u}{\partial y} = y^3 (x - y)$$

$$y \frac{\partial u}{\partial y} = x y^4^3 - y^2 u^3$$

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = y^2 u^3$$

Q.5

$$u = x^y$$

Q.6

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) \right)$$

$$\frac{\partial u}{\partial y} = x^y (\log x)$$

Q.7

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) = y x^{y-1} (\log x) + x^{y-1}$$

$$= x^{y-1} [y \log x + 1]$$



$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) \right) =$$

$$= (y-1)x^{y-2} [y \log x + 1] + x^{y-1} \left[ -\frac{y}{x} \right]$$

$$= y^2 x^{y-2} \log x - x^{y-2} + x^{y-2} y$$

$$\frac{\partial u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) \right)$$

$$\frac{\partial u}{\partial x} = yx^{y+1}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) = x^{y+1} + \cancel{y(y+1)x^{y+1} \log x} + yx^{y+1} \log x$$

$$x^{y+1} (1 + y \log x)$$

$$\frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) \right) =$$

$$= (y-1)x^{y-2} (1 + y \log x) + x^{y+1} \left( \frac{y}{x} \right)$$

$$= yx^{y-2} - x^{y-2} y \log x + x^{y-2} y \quad \text{--- (2)}$$

$$= yx^{y-2} + y^2 x^{y-2} \log x - x^{y-2} - x^{y-2} y \log x$$

from (1) and (2)

$$+ x^{y-2} y \quad \text{--- (2)}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial^3 u}{\partial x^3 \partial y}$$



Q. If  $x^x y^y z^z = c$

s.t. at  $x = y = z = 2$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = - \left[ x \log(x) \right]^{-1}$$

Practically, diff. w.r.t.  $y$

$$x \log x + y \log y + z \log z = \log c \quad \text{--- (1)}$$

$$(1 + \log y) + (1 + \log z) \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = - \frac{(1 + \log y)}{(1 + \log z)}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = - \frac{(1 + \log y)}{(1 + \log z)^2}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = - (1 + \log y) \left[ \frac{-1}{(1 + \log z)^2} \right] \times \frac{1}{z} \times \frac{\partial z}{\partial x}$$

P. diff. (1) w.r.t.  $x$

$$(1 + \log x) + (1 + \log z) \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = - \frac{(1 + \log x)}{(1 + \log z)}$$



$$\boxed{\log e = 1}$$

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$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{-(1+\log y)}{(1+\log z)^3} \times \frac{1}{3} (1+\log x)$$

at  $x=y=z$ .

$$-\left( \frac{1}{(1+\log x)} \times \frac{1}{2} \right)$$

$$-\left[ \log(e^x) x \right]^{-1}$$

Q=

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\text{S.T.} \quad \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{(x+y+z)^2}$$

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) u$$

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3x^2 - 3yz)$$

$$\frac{\partial u}{\partial y} = \frac{3z^2 - 3xy}{(x^3 + y^3 + z^3 - 3xyz)}$$

$$\frac{\partial u}{\partial z} = \frac{3x^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$



$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 + 3y^2 + 3z^2 - 3xy - 3yz - 3zx}{(x^3 + y^3 + z^3 - 3xyz)}$$

$$= \frac{3(x^3 + y^3 + z^3 - 3xy - yz - zx)}{(x^3 + y^3 + z^3 - 3xy - yz - zx)(x+y+z)}$$

$$= \frac{3}{(x+y+z)}$$

$$\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left( \frac{3}{x+y+z} \right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

\* if  $x = r \cos \theta$ ,  $y = r \sin \theta$

$$(1) \left( \frac{\partial}{\partial x} \right)_{y=\text{const.}}, \left( \frac{\partial}{\partial y} \right)_{x=\text{const.}}$$

$$\left( \frac{\partial}{\partial r} \right)_{\theta=\text{const.}}, \left( \frac{\partial}{\partial \theta} \right)_{r=\text{const.}}$$



$$\textcircled{2} \left( \frac{\partial z}{\partial x} \right)_y = \text{const}, \quad \left( \frac{\partial z}{\partial y} \right)_x = \text{const}$$

$$\left( \frac{\partial y}{\partial z} \right)_{x=\text{const}} \quad \left( \frac{\partial x}{\partial z} \right)_{y=\text{const}}$$

Q.

$$z = (x \log(x+r)) - r$$

where  $r^2 = x^2 + y^2$  — (1)

$$\text{p.T} \quad \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = \frac{1}{x+r}$$

$$\frac{\partial z}{\partial x} = \log(x+r) + \frac{x}{(x+r)} \left( 1 + \frac{\partial r}{\partial x} \right) - \frac{\partial r}{\partial x}$$

Partially diff. eqn (1) w.r.t.  $x = x$ 

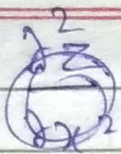
$$2x \frac{\partial r}{\partial x} = 2x$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$\frac{\partial z}{\partial x} = \log(x+r) + \frac{x}{x+r} + \frac{x^2}{r(x+r)} - \frac{x}{r}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x+r} \left( 1 + \frac{x}{r} \right) + \left( \frac{r \cdot x \cdot (1+r) - x(r+x)}{(x+r)^2} \right) + \left[ \frac{2xy \cdot (r(x+r)) - \dots}{(x+r)^2} \right]$$





$$\frac{\partial^2 z}{\partial x^2} = \log(x+r)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{(x+r)} \left( 1 + \frac{\partial r}{\partial x} \right)$$

$$= \frac{1}{(x+r)} \left( 1 + \frac{x}{r} \right)$$

$$= \frac{1}{r}$$

$$\frac{\partial z}{\partial y} = \frac{r}{x+r} - \frac{r}{r} = \frac{r}{x+r} - 1$$

$$\frac{\partial z}{\partial y} = \frac{r}{x+r} - 1$$

$$= \frac{(1+r)y - y}{(x+r)r}$$

$$= \frac{-y}{x+r}$$

$$\frac{\partial^2 z}{\partial y^2} = - \left[ \frac{(1+r)(1) - y \left( \frac{\partial r}{\partial y} \right)}{(x+r)^2} \right]$$



$$= - \left[ \frac{cx + r - \frac{y^2}{r}}{(x+r)^2} \right]$$

$$= - \left[ \frac{rx + r^2 - y^2}{r(x+r)} \right]$$

$$= - \left[ \frac{rx + r^2}{r(x+r)^2} \right]$$

$$\frac{\partial^2 z}{\partial y^2} = - \frac{r}{r(x+r)}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r} - \frac{r}{r(x+r)}$$

$$= \frac{x+r-r}{r(x+r)}$$

$$= \frac{1}{x+r}$$



Q. If  $\theta = r^n e^{-\frac{r^2}{4t}}$

Find the value of  $n$  if  $\theta = r^n e^{-\frac{r^2}{4t}}$

that will make

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial \theta}{\partial r} \right] = \frac{\partial \theta}{\partial t}$$

$$\frac{\partial \theta}{\partial r} = \left( n r^{n-1} e^{-\frac{r^2}{4t}} + r^n e^{-\frac{r^2}{4t}} \times \frac{-2r}{4t} \right)$$

$$= e^{-\frac{r^2}{4t}} \left[ n r^{n-1} - \frac{r^{n+1}}{2t} \right]$$

$$r^2 \frac{\partial \theta}{\partial r} = e^{-\frac{r^2}{4t}} \left[ n r^{n+1} - \frac{r^{n+3}}{2t} \right]$$

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right) =$$

$$= e^{-\frac{r^2}{4t}} \left( -\frac{2r}{4t} \right) \left[ n r^{n+1} - \frac{r^{n+3}}{2t} \right]$$

$$+ e^{-\frac{r^2}{4t}} \left[ n(n+1) r^n - \frac{(n+3)}{2t} r^{n+2} \right]$$



$$= -\frac{e^{-\frac{r^2}{4t}}}{2t} r \left[ nr^{n+1} - \frac{r^{n+3}}{2t} \right] + e^{-\frac{r^2}{4t}} \left[ n(n+1) r^{n+2} - \frac{(n+3) r^{n+2}}{2t} \right]$$

$$= -\frac{e^{-\frac{r^2}{4t}}}{2t} \left[ nr^{n+1} - \frac{r^{n+3}}{2t} \right] + e^{-\frac{r^2}{4t}} \left[ n(n+1) r^{n+2} - \frac{(n+3) r^{n+2}}{2t} \right]$$

$$\frac{\partial \phi}{\partial t} =$$



## \* Euler's Theorem :

### (1) Homogeneous function :

$$\begin{aligned} f(tx, ty) &= f(tx, ty) \\ &= t^n f(x, y) \end{aligned}$$

→ If  $u$  is the homogeneous  $f^n$  of  $x$  and  $y$  of degree  $n$  then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

$$\text{let } u = x^n \cdot f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right)$$

$$x \frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) + x^{n-1} y f'\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right) \times \left(\frac{1}{x}\right)$$

$$y \frac{\partial u}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right) \quad \text{--- (2)}$$

add eqn (1) and (2)



↓

$$\frac{x \partial u}{\partial x} + \frac{y \partial u}{\partial y} = n x^n f(y/x)$$

$$= n u,$$

(Cont)

Imp

Q If  $u = \sin^{-1} \left[ \frac{x + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}} \right]$

by Euler's theorem.

P.T  $\frac{x \partial u}{\partial x} + \frac{y \partial u}{\partial y} + \frac{z \partial u}{\partial z} + 3 \tan u = 0$

$$v = \sin u = \frac{x + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}}$$

Now  $v$  is Homogeneous fn of  $x, y$  and  $z$  of  
order  $-3$ .

$$\frac{x \partial v}{\partial x} + \frac{y \partial v}{\partial y} + \frac{z \partial v}{\partial z} = -3v$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} + z \cos u \frac{\partial u}{\partial z}$$

$$= -3 \sin u.$$

$$\frac{x \partial u}{\partial x} + \frac{y \partial u}{\partial y} + \frac{z \partial u}{\partial z} + 3 \tan u = 0$$



$$Q. \quad \theta = t^n e^{-\frac{r^2}{4t}}$$

$$\frac{\partial \theta}{\partial r} = t^n \left[ e^{-\frac{r^2}{4t}} \times -\frac{2r}{4t} \right]$$

$$\frac{\partial \theta}{\partial r} = t^n \left[ e^{-\frac{r^2}{4t}} \times -\frac{r}{2t} \right]$$

$$= -\frac{t^{n-1}}{2} (e^{-r^2/4t}) \cdot r$$

$$\frac{1}{r^2} \left( \frac{\partial \theta}{\partial r} \right) = -\frac{t^{n-1}}{2} e^{-r^2/4t} \times r^{-1}$$

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right) = -\frac{t^{n-1}}{2} \left[ 3r^2 e^{-r^2/4t} - e^{-r^2/4t} \frac{r^2}{2t} \right]$$

$$\frac{1}{r^2} \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right) \right) = -\frac{t^{n-1}}{2} e^{-r^2/4t} \left[ 3 - \frac{r^2}{2t} \right]$$

$$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t} \cdot \frac{r^2}{4t^2}$$

$$= t^{n-1} e^{-r^2/4t} \left( n + \frac{r^2}{4t^2} \right)$$



$$-\frac{t^n e^{-\frac{r^2}{4t}}}{2} \left[ 3 - \frac{r^2}{2t} \right] = t^n e^{-\frac{r^2}{4t}} \left( n + \frac{r^2}{4t} \right)$$

$$-\frac{3}{2} + \frac{r^2}{4t} = n + \frac{r^2}{4t}$$

$$\boxed{n = -\frac{3}{2}}$$

Q. Verify Euler's theorem for the  $u$

o dem

$$u = \sin^{-1} \frac{x}{y} + \tan^{-1} \left( \frac{y}{x} \right)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \times \left( \frac{1}{y} \right) + \frac{1}{1 + \frac{y^2}{x^2}} \times \left( -\frac{y}{x^2} \right)$$

$$x \frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \left( -\frac{x}{y^2} \right) + \frac{1}{1 + \frac{y^2}{x^2}} \left( \frac{1}{x} \right)$$

$$y \frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} \times \left( -\frac{y}{y^2} \right) + \frac{y}{1 + \frac{y^2}{x^2}} \left( \frac{1}{x} \right)$$



$$\frac{x^2+y^2}{x}$$

$$y \frac{\partial y}{\partial y} = \frac{-x}{\sqrt{y^2+x^2}} + \frac{xy}{x^2+y^2} \quad \text{--- (3)}$$

$$Q. \quad u = \log \left( \frac{x^2+y^2}{x+y} \right)$$

$$v = e^u = \frac{x^2+y^2}{x+y}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = v$$

$$v = e^u$$

$$\frac{\partial v}{\partial x} = e^u \frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = e^u \frac{\partial u}{\partial y}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = v$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = \frac{e^u}{e^u}$$



Que:  $u = \log(\tan x + \tan y + \tan z)$

P.T  $\sin x \frac{\partial u}{\partial x} + \sin y \frac{\partial u}{\partial y} + \sin z \frac{\partial u}{\partial z} = 2$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin x \frac{\partial u}{\partial x} = \frac{2 \tan x}{\tan x + \tan y + \tan z} \quad \text{--- (1)}$$

$$\sin y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z} \quad \text{--- (2)}$$

$$\sin z \frac{\partial u}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z} \quad \text{--- (3)}$$

add (1), (2) and (3)

$$\sin x \frac{\partial u}{\partial x} + \sin y \frac{\partial u}{\partial y} + \sin z \frac{\partial u}{\partial z} = 2$$

Q- if  $Z$  is a Homogeneous fn of degree  $n$

P.T (i)  $x \frac{\partial^2 Z}{\partial x^2} + y \frac{\partial^2 Z}{\partial x \partial y} = (n+1) \frac{\partial Z}{\partial x}$

(ii)  $x \frac{\partial^2 Z}{\partial y \partial x} + y \frac{\partial^2 Z}{\partial y^2} = (n+1) \frac{\partial Z}{\partial y}$



For

$$\#4 \text{ continuity} = \boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

$$(iii) \quad \frac{x^2 \partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n+1)z$$

(i) by Euler's theorem

$$\frac{x \partial z}{\partial x} + \frac{y \partial z}{\partial y} = nz \quad \text{--- (1)}$$

P.D. w.r.t x

x. ~~z~~

$$\frac{\partial z}{\partial x} + x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x}$$

$$\frac{x \partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = (n+1) \frac{\partial z}{\partial x} \quad \text{--- (2)}$$

(ii) P.D. w.r.t y

$$x \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial y} + y \frac{\partial^2 z}{\partial y^2} = n \frac{\partial z}{\partial y}$$

$$\frac{x \partial^2 z}{\partial y \partial x} + \frac{y \partial^2 z}{\partial y^2} = (n+1) \frac{\partial z}{\partial y} \quad \text{--- (3)}$$

$$(iii) \quad e^x \textcircled{2} x x + e^y \textcircled{3} x y$$



$$\frac{x^2 \partial^2 z}{\partial x^2} + xy \frac{\partial^2 z}{\partial x \partial y} + xy \frac{\partial^2 z}{\partial y \partial x} + y^2 \frac{\partial^2 z}{\partial y^2} = (n+1)x \frac{\partial z}{\partial x} + (n+1)y \frac{\partial z}{\partial y}$$

because  $\nabla^2 z$  is continuous

$$\textcircled{1} \quad \frac{x^2 \partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (n+1)x^2$$

Exmp. (c)

Que

By using E.T

RT if  $u = \tan^{-1}(x^2 + y^2)$

then  $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$

$\frac{y}{x^2+y^2} / (\frac{y}{x^2+y^2} - 1)$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= 2 \sin u \cos 3u$$

$$u = \tan^{-1}(x^2 + y^2)$$

$$v = \tan u = (x^2 + y^2) \quad \text{of order } \textcircled{2}$$



$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

$$\begin{aligned} \rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= 2 \sin u \cos u \\ &= \sin 2u = g(u) \end{aligned}$$

$$\rightarrow x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \frac{g(u) - 1}{g(u)(g'(u) - 1)}$$

$$= \sin 2u [2 \cos 2u - 1]$$

$$= \sin 2u [2(\cos^2 u - \sin^2 u) - 1]$$

$$= \sin 2u [2(\cos^2 u - (1 - \cos^2 u)) - 1]$$

$$= \sin 2u (2(2\cos^2 u - 1) - 1)$$

$$= 2 \sin u [4 \cos^2 u - 3 \cos u]$$

$$= 2 \sin u \cos 3u$$



Que:

$$Q = Z = \frac{x^n}{w} f\left(\frac{y}{x}\right) + y \frac{-n}{t} \phi\left(\frac{x}{y}\right)$$

$$P.T \quad x^2 \frac{\partial^2 Z}{\partial x^2} + 2xy \frac{\partial^2 Z}{\partial x \partial y} + y^2 \frac{\partial^2 Z}{\partial y^2} =$$

$$+ x \frac{\partial Z}{\partial x} + y \frac{\partial Z}{\partial y} = n^2 Z$$

$$Z = u + v$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu \quad \text{--- (1)}$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = -nv \quad \text{--- (2)}$$

$$x \frac{\partial Z}{\partial x} = x \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x}$$

$$y \frac{\partial Z}{\partial y} = y \frac{\partial u}{\partial y} + y \frac{\partial v}{\partial y}$$

$$x \frac{\partial Z}{\partial x} + y \frac{\partial Z}{\partial y} = nu - nv$$

Partial diff. w.r. to x

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n+1)u \quad \text{--- (3)}$$

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = -n(n+1)v \quad \text{--- (4)}$$



adding (3) and (4)

$$x^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} \right) + 2xy \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \right) + y^2 \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \right) =$$

$$= n(n+1)u + (n)(n+1)u$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \left( \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} \right) =$$

$$= n(n+1)u + n(n+1)u$$

$$= (n^2 - n)u + (n^2 + n)u$$

$$= n$$

$$\frac{x^2 \partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + \frac{2xz}{x} + \frac{y^2 z}{y}$$

$$= n^2 z - n(4-u) + n(4-u)$$



Ques:

if  $u = \sin^{-1} \left( \frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$

Prove

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y}$$

$$= \frac{\sin u \cos 2u}{4 \cos^3 u}$$

$$u = \sin^{-1} u = \frac{x+y}{\sqrt{x} + \sqrt{y}}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{\sin u}{2}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\tan u}{2}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \sec u (\sec u - 1)$$

$$= \frac{\tan u}{2} \left( \frac{\sec^2 u}{2} - 1 \right)$$

$$= \frac{\sin u}{2 \cos u} \left( \frac{1}{2 \cos^2 u} - 1 \right)$$

$$2 \cos^2 u = \frac{1 - \cos 2u}{2}$$



$$= \frac{-\sin y \cos 3y}{4 \cos 3y}$$

\* Total differentiation

$$\frac{dy}{dt} = \frac{\partial u}{\partial x} \times \frac{dx}{dt} + \frac{\partial u}{\partial y} \times \frac{dy}{dt}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

Q.  $u = u(y-z, z-x, x-y)$

P.T  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

$$u = u(r, s, t)$$

$$r = y-z, \quad s = z-x, \quad t = x-y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \times \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \times \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} \times 0 + \frac{\partial u}{\partial s} \times (-1) + \frac{\partial u}{\partial t} \times 1$$

⊖



$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial y}{\partial s} \times \frac{\partial s}{\partial t} + \frac{\partial y}{\partial t} \times \frac{\partial t}{\partial t}$$

$$= \frac{\partial y}{\partial x} \times 1 + \frac{\partial y}{\partial s} \times 0 + \frac{\partial y}{\partial t} \times (-1) \quad \text{--- (2)}$$

$$\frac{\partial y}{\partial z} = \frac{\partial y}{\partial x} \times \frac{\partial x}{\partial z} + \frac{\partial y}{\partial s} \times \frac{\partial s}{\partial z} + \frac{\partial y}{\partial t} \times \frac{\partial t}{\partial z}$$

$$= \frac{\partial y}{\partial x} (-1) + \frac{\partial y}{\partial s} \times 1 + \frac{\partial y}{\partial t} \times 0 \quad \text{--- (3)}$$

add (1), (2) and (3)

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial y}{\partial z} = 0$$

Q.

$$z = f(x, y)$$

$$x = e^u \cos v, \quad y = e^u \sin v$$

$$\text{P.T.} \quad y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v} = e^{2u} \frac{\partial z}{\partial y}$$

$$\left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 = e^{-2u} \left[ \left( \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial v} \right)^2 \right]$$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} (e^4 \cos u) + \frac{\partial z}{\partial y} \times e^4 \sin u$$

$$y \frac{\partial z}{\partial y} = xy \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x} (-e^4 \sin u) \times \frac{\partial z}{\partial y} \times e^4 \cos u$$

$$x \frac{\partial z}{\partial v} = -xy \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} \quad \text{--- (2)}$$

Adding (1) and (2)

$$y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial v} = (x^2 + y^2) \frac{\partial z}{\partial y}$$

$$= e^{2u} \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial u} = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$$



$$x^2 \left( \frac{\partial z}{\partial x} \right)^2 + y^2 \left( \frac{\partial z}{\partial y} \right)^2 + 2 \left( \frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} \right) \quad (1)$$

$$\frac{\partial z}{\partial u} = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$$

$$\left( \frac{\partial z}{\partial u} \right)^2 = x^2 \left( \frac{\partial z}{\partial x} \right)^2 + y^2 \left( \frac{\partial z}{\partial y} \right)^2 + 2xy \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \quad (2)$$

$$\frac{\partial z}{\partial v} = -y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y}$$

$$\left( \frac{\partial z}{\partial v} \right)^2 = y^2 \left( \frac{\partial z}{\partial x} \right)^2 + x^2 \left( \frac{\partial z}{\partial y} \right)^2 - 2xy \frac{\partial z}{\partial x} \frac{\partial z}{\partial y}$$

add (1) and (2)

$$\left( \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial v} \right)^2 = e^{2u} \left( \frac{\partial z}{\partial x} \right)^2 + e^{2u} \left( \frac{\partial z}{\partial y} \right)^2$$

$$e^{-2u} \left[ \left( \frac{\partial z}{\partial u} \right)^2 + \left( \frac{\partial z}{\partial v} \right)^2 \right] = \left[ \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right]$$



Q. If  $u = u(e^{y-z}, e^{z-x}, e^{x-y})$

Find  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$



Q. if  $\phi(x-az, y-bz) = 0$

Show that

$$ap + bq = c$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

~~$\phi(x, y, z) = 0$~~   $\phi(x, z) = 0$

$$x = (x - az), \quad z = (y - bz)$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$0 = \frac{\partial \phi}{\partial x} = c - a \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = - \frac{b \partial z}{\partial x}$$

$$0 = \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial x} \cdot (c - a \frac{\partial z}{\partial x}) + \frac{\partial \phi}{\partial z} \cdot (-b \frac{\partial z}{\partial x})$$

$$0 = c \frac{\partial \phi}{\partial x} - a \frac{\partial \phi}{\partial x} \cdot \frac{\partial z}{\partial x} - b \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{c \partial \phi / \partial x}{a \frac{\partial \phi}{\partial x} + b \frac{\partial \phi}{\partial z}}$$



$$\frac{a \partial z}{\partial x} - \frac{ac \frac{\partial \phi}{\partial r}}{\frac{a \partial \phi}{\partial r} + b \frac{\partial \phi}{\partial s}} \quad \text{--- (1)}$$

~~By~~ =

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial s} \frac{\partial s}{\partial y}$$

$$\frac{\partial r}{\partial y} = -\frac{a \partial z}{\partial y}, \quad \frac{\partial s}{\partial y} = c - b \frac{\partial z}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \left( -\frac{a \partial z}{\partial y} \right) + \frac{\partial \phi}{\partial s} \left( c - b \frac{\partial z}{\partial y} \right)$$

$$\frac{\partial \phi}{\partial y} = -\frac{a \partial \phi}{\partial r} \frac{\partial z}{\partial y} + c \frac{\partial \phi}{\partial s} - b \frac{\partial \phi}{\partial s} \frac{\partial z}{\partial y}$$

~~By~~

$$0 = -\frac{a \partial \phi}{\partial r} \frac{\partial z}{\partial y} + c \frac{\partial \phi}{\partial s} - b \frac{\partial \phi}{\partial s} \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{c \frac{\partial \phi}{\partial s}}{\frac{a \partial \phi}{\partial r} + b \frac{\partial \phi}{\partial s}} \quad \text{--- (2)}$$



$$b \frac{\partial z}{\partial y} = \frac{bc \frac{\partial \phi}{\partial s}}{a \frac{\partial \phi}{\partial r} + b \frac{\partial \phi}{\partial s}} \quad \text{--- (2)}$$

Adding (1) and (2)

$$a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} =$$

$$c \left[ a \frac{\partial \phi}{\partial r} + b \frac{\partial \phi}{\partial s} \right] \frac{1}{\left[ a \frac{\partial \phi}{\partial r} + b \frac{\partial \phi}{\partial s} \right]}$$

$$= c$$

Ex 10

Ques. If  $u = f(r)$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$

P.T

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{f'(r)}{r}$$

$$r^2 = x^2 + y^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$r^2 = x^2 + y^2$$

$$2r \frac{dr}{dx} = 2x$$



$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial u}{\partial x} = \left( \frac{\partial u}{\partial r} \cdot \frac{x}{r} \right)$$

$$= \cancel{f'(r)} f'(r) \left( \frac{x}{r} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = \left( f''(r) \times \frac{\partial r}{\partial x} \right) \left( \frac{x}{r} \right) + f'(r) \left( \frac{1 \cdot r - x \frac{\partial r}{\partial x}}{r^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r) \frac{x^2}{r^2} + f'(r) \left( \frac{r^2 - x^2}{r^3} \right)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r) \frac{x^2}{r^2} + f'(r) \frac{y^2}{r^3}$$

similarly.

$$\frac{\partial^2 u}{\partial y^2} = f''(r) \frac{y^2}{r^2} + f'(r) \frac{x^2}{r^3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{f'(r)}{r}$$



Q.  $u = u(x, y)$

$$x = s \cos \alpha - t \sin \alpha$$

$$y = s \sin \alpha + t \cos \alpha$$

P.T  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}$

Proof

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \times \cos \alpha + \frac{\partial u}{\partial y} \times \sin \alpha$$

$$\frac{\partial}{\partial s} = \cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y}$$

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s} \left( \frac{\partial u}{\partial s} \right) =$$

$$= \left[ \cos^2 \alpha \frac{\partial^2 u}{\partial x^2} + 2 \sin \alpha \cos \alpha \frac{\partial^2 u}{\partial x \partial y} + \right.$$

$$\left. + \sin^2 \alpha \frac{\partial^2 u}{\partial y^2} \right] \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial t}$$



L-Ros.  $\frac{d}{dt}$  - Remain  $\rightarrow$

Variable  $\rightarrow$  maximum  $\rightarrow$  minimum  $\rightarrow$  free

5,

185

185

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} (-\sin \alpha) + \frac{\partial y}{\partial y} (\cos \alpha)$$

$$\frac{\partial}{\partial t} = -\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right)$$

$$\frac{\partial^2 u}{\partial t^2} = \sin^2 \alpha \frac{\partial^2 u}{\partial x^2} - 2 \sin \alpha \cos \alpha \frac{\partial^2 u}{\partial x \partial y} + \cos^2 \alpha \frac{\partial^2 u}{\partial y^2} \quad (3)$$

(1) (2)

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Ques: Pf  $z = z(x, y) = z(u, v)$

and  $u = lx + my$

$v = ly - mx$

To that  $\rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2) \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} \times l + \frac{\partial z}{\partial v} (-m)$$



$$= l \frac{\partial z}{\partial u} - m \frac{\partial z}{\partial v}$$

$$\frac{\partial}{\partial x} = l \frac{\partial}{\partial u} - m \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = l^2 \frac{\partial^2 z}{\partial u^2} - m l \frac{\partial^2 z}{\partial u \partial v} - m l \frac{\partial^2 z}{\partial v \partial u} + m^2 \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2} = l^2 \frac{\partial^2 z}{\partial u^2} - 2ml \frac{\partial^2 z}{\partial u \partial v} + m^2 \frac{\partial^2 z}{\partial v^2} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} (m) + \frac{\partial z}{\partial v} (l)$$

$$\frac{\partial}{\partial y} = m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = m^2 \frac{\partial^2 z}{\partial u^2} + l m \frac{\partial^2 z}{\partial u \partial v} + m l \frac{\partial^2 z}{\partial v \partial u} + l^2 \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = m^2 \frac{\partial^2 z}{\partial u^2} + 2ml \frac{\partial^2 z}{\partial u \partial v} + l^2 \frac{\partial^2 z}{\partial v^2} \quad \text{--- (2)}$$

Adding eq<sup>n</sup> (1) and (2)



$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + m^2 \frac{\partial^2 z}{\partial v^2} + m^2 \frac{\partial^2 z}{\partial u^2} + l^2 \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} (l^2 + m^2) + \frac{\partial^2 z}{\partial v^2} (m^2 + l^2)$$

$$= (l^2 + m^2) \left( \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right) \underline{\Delta y}$$

~~X~~ Q-3  $z = f(x, y) = 0$   
 $f$  is in  $C^2$  form

$$\frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y}$$

Ques: Find  $\frac{dy}{dx}$  if  $(\cos x)^y = (\sin y)^x$

$$(\cos x)^y - (\sin y)^x = 0$$

$$\frac{\partial f}{\partial x} = y \cos x^{y-1} (\sin x) - (\sin y)^x \log(\sin y)$$

$$\frac{\partial f}{\partial y} = (\cos x)^y \log(\cos x) - x \sin y^{x-1} \cos y$$

$$\frac{dy}{dx} = \frac{[y \tan x + \log(\sin y)] (\cos x)^y}{[\log(\cos x) - x \cot y] (\cos x)^y}$$



$$= \frac{y \tan x + \log(\sin y)}{\log(\cos x - x \cot y)}$$



Que if  $f(x, y) = 0$  &  $\phi(y, z) = 0$

P.T

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

$$\frac{dy}{dx} = \left( \frac{-\partial f / \partial x}{\partial f / \partial y} \right)$$

$$\frac{dz}{dy} = - \frac{\partial \phi / \partial y}{\partial \phi / \partial z}$$

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = \frac{\partial f / \partial x \cdot \partial \phi / \partial y}{\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}}$$

$$\frac{\partial z}{\partial x} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$



$$u = x, y$$

Q. If  $u = x \log y$ .

$$x^3 + y^3 + 3xy = 1 \quad \text{--- (1)}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$$

Total  
derivative

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

diff (1) w.r. to  $x$

~~$$3x^2 + 3y^2 \frac{dy}{dx} + 3x \frac{dy}{dx}$$~~

$$3x^2 + 3y^2 \frac{dy}{dx} + 3 \left( y + x \frac{dy}{dx} \right) = 0$$

$$\frac{dy}{dx} = \frac{-x^2 - 3y}{3x + 3y^2}$$

$$\frac{dy}{dx} = \frac{-x^2 - y^2}{x^2 + y^2}$$

$$\frac{dy}{dx} = (\log y) - \frac{x}{y} \left( \frac{x^2 + y^2}{x + y^2} \right)$$



## \* errors and approximation:

$$\rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$$

$$\frac{dy}{dx} \sim \frac{\delta y}{\delta x}$$

$$y = f(x)$$

$$\log y = \log f(x)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Hence

$$\frac{1}{y} \frac{\delta y}{\delta x} = \frac{f'(x)}{f(x)}$$

$$\boxed{\delta y = y \frac{f'(x)}{f(x)} \delta x}$$

Here  $\delta x$  is called absolute error in  $x$   
 $\delta y$  " " " " " "  $y$

and  $\frac{\delta x}{x}$  is called relative error in  $x$ .

$\frac{\delta x}{x} \times 100$  is called % error



Ques: diameter and altitude of a cone in the shape of right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the max possible error in the value computed for volume and lateral surface.

Sol<sup>n</sup>  $d = 4 \text{ cm}$ ,  $h = 6 \text{ cm}$   
 $\delta d = 0.1 \text{ cm}$   $\delta h = 0.1 \text{ cm}$

$$V = \pi r^2 h$$

$$= \frac{\pi d^2 h}{4}$$

$$\delta V = \frac{\pi}{4} [2d \delta d h + d^2 \delta h]$$

$$\delta V = 5.033 \text{ cm}^3$$

$$S = 2\pi r h$$

$$= \pi d h$$

$$\delta S = \pi (\delta d h + d \delta h)$$

$$= 3.146 \text{ cm}^2$$



Q1. If KE ( $T = \frac{1}{2}mv^2$ ) find approximately the change in  $T$  as ~~m~~  $m$  changes from 49 to 49.5 and  $v$  changes from 1600 to 1590

$$\frac{\partial T}{\partial t} = 0.05$$

$$m = 49, \quad m + \delta m = 49.5$$

$$v = 1600, \quad v + \delta v = 1590$$

$$\delta m = 0.5, \quad \delta v = -10$$

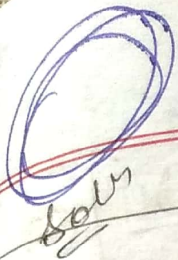
$$T = \frac{1}{2}mv^2$$

$$\delta T = \frac{1}{2} [\delta m v^2 + 2m v \delta v]$$

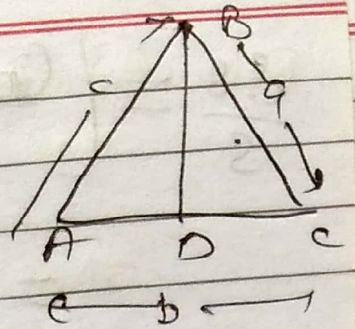
$$\delta T = -144000$$

Q2: The area of triangle is calculated from the angles  $A$ ,  $B$ ,  $C$  and the side  $a$ . If  $\delta A$  is the error in measuring  $A$  show that the relative error in the area is approximately  $\frac{\sin C \delta A}{\sin A \sin(A+C)}$





$$S = \frac{1}{2} AC \times BD$$



$$\sin C = \frac{BD}{a}$$

$$BD = a \sin C$$

$$S = \frac{1}{2} ab \sin C$$

(Remove 'a')

we know

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$a = \frac{b \sin A}{\sin B}$$

$$S = \frac{1}{2} \frac{b^2 \sin A \sin C}{\sin B}$$

$$S = \frac{1}{2} \frac{b^2 \sin A \sin C}{\sin(A+C)}$$

$$\log S = \log \frac{1}{2} b^2 \sin C + \log \sin A - \log \sin(A+C)$$

$$- \log \sin(A+C)$$

$$\frac{1}{S} = \frac{1}{\sin A} \cos A - \frac{\cos(A+C)}{\sin(A+C)}$$

having 1 cm

~~cos A~~



$$\log_e x = \log_{10} \frac{10}{x}$$

$$\frac{\delta s}{s} = \left[ \frac{\cos A \sin(A+C) - \sin A \cos(A+C)}{\sin A \sin(A+C)} \right] \delta A$$

$$= \frac{\sin C}{\sin A \sin(A+C)} \delta A$$

Q. what error in the common log. of a no. will be produced by an error of 1% in number

Sol<sup>n</sup>:

$$\frac{\partial x}{x} \times 100 = 1$$

$$y = \log_{10} x$$

$$y = \log_{10} e \log_e x$$

$$\delta y = \log_{10} e \left[ \frac{1}{x} \delta x \right]$$

$$\delta y \times 100 = \log_{10} e \left[ \frac{\partial x}{x} \times 100 \right]$$

$$\delta y = \frac{\log_{10} e}{100}$$

$$\delta y = \frac{0.4342}{100}$$

$$= 0.004342$$



Imp: mark

Ques The time period  $T$  of a simple pendulum of length  $l$  is given

$$T = 2\pi \sqrt{\frac{l}{g}}$$

$$(g - \delta g) =$$

find (i) the error

(ii) % error

(iii) made in computing  $T$  by using

$$l = 2 \text{ ft}$$

$$g = 32 \text{ ft/sec}^2$$

if the true values are

$$l = 1.95 \text{ ft}$$

$$g = 32.2 \text{ ft/sec}^2$$

$$\delta l = 0.05, \quad \delta g = -0.2$$

$$\log T = \log 2\pi + \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\frac{1}{T} \delta T = \frac{1}{2l} \delta l - \frac{1}{2g} \delta g$$

$$\frac{\delta T}{T} = \frac{0.05}{2 \times 1.95} + \frac{(0.02)}{2 \times 32.2}$$

$$\frac{\delta T}{T} = 0.0156$$

$$\frac{\delta T}{T} \times 100 = 1.5625 \%$$

~~$\delta T$~~

$$\text{error} = \delta T = 0.0246$$



Ques

The Period of a Simple Pendulum

$$T = 2\pi \sqrt{\frac{l}{g}}$$

find the max. error in  $T$  due to the possible errors of  $1\%$  in  $l$  and  $2.5\%$  in  $g$

$$\log T = \log 2\pi + \frac{1}{2} \log l + \frac{1}{2} \log g$$

$$\frac{1}{T} \delta T = \frac{1}{2l} \delta l + \frac{1}{2g} \delta g$$

$$\frac{\delta T}{T} \times 100 = \frac{1}{2l} \delta l \times 100 + \frac{1}{2g} \delta g \times 100$$

← maximum error possible

$$= \frac{1}{2} \times 1 + \frac{1}{2} \times 2.5$$

$$= 1.75\%$$

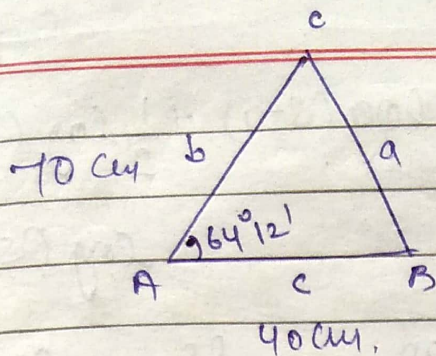
Ques

Two sides of a triangle are  $70\text{ cm}$  and  $40\text{ cm}$  in length. The angle b/w them  $64^\circ 12'$  but the angle is measured by mistake as  $65^\circ$  what is the error in the calculation of length of third side



$$\sqrt{s(s-a)(s-b)(s-c)}$$

cm  $\rightarrow$  Radian.  $197$



$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$2a \delta a = + 2bc (\sin A) \delta A$$

$$\delta A = 65^\circ - \frac{64.2 \times 5}{5}$$

$$= 65^\circ - 64.2 = 0.8^\circ$$

$$= 0.8 \times \frac{\pi}{180}$$

$$2a \delta a = 2 \times 70 \times 40 \times \sin(64.2) \times \left(0.8 \times \frac{\pi}{180}\right)$$

$$\delta a = 0.55 \text{ cm}$$

Prob.

Q. with the usual meaning of  $a, b, c, s$  if  $\Delta$  is the area of triangle P.T the error in  $\Delta$  resulting from the small error in the measurement of  $c$  is given by

$$\frac{\delta \Delta}{\Delta} = \frac{\Delta}{4} \left[ \frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right] \delta c$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{a+b+c}{2}$$



$$\log \Delta = \frac{1}{2} \log s + \frac{1}{2} \log (s-a) + \frac{1}{2} \log (s-b) + \frac{1}{2} \log (s-c)$$

$$\therefore \frac{\delta \Delta}{\Delta} = \frac{\delta s}{2s} + \frac{\delta s}{2(s-a)} + \frac{\delta s}{2(s-b)} + \frac{\delta s - \delta c}{2(s-c)}$$

$$\Delta s = \frac{\delta c}{2}$$

$$\frac{\delta \Delta}{\Delta} = \frac{\delta c}{4s} + \frac{\delta c}{4(s-a)} + \frac{\delta c}{4(s-b)} + \frac{\frac{\delta c}{2} - \delta c}{2(s-c)}$$

$$\frac{\delta \Delta}{\Delta} = \frac{\delta c}{4} \left[ \frac{1}{s} + \frac{1}{(s-a)} + \frac{1}{(s-b)} - \frac{1}{(s-c)} \right]$$

Ques!

In estimating the no. of bricks in a pile to be 5m x 10m x 5m Count of bricks is taken as 100 bricks/m<sup>3</sup> find the error in cost when the measuring tape is stretched 2% beyond its standard length the cost of bricks is Rs. 2000 per 1000 bricks

$$V = xyz =$$

$$\log V = \log x + \log y + \log z$$



$$\frac{1}{V} dV = \frac{1}{x} \delta x + \frac{1}{y} \delta y + \frac{1}{z} \delta z$$

$$\frac{\delta V}{V} \times 100 = \frac{1}{x} \delta x \times 100 + \frac{1}{y} \delta y \times 100 + \frac{1}{z} \delta z \times 100$$

$$\frac{\delta V}{V} \times 100 = 60$$

$$\delta V = \frac{60 \times 5 \times 10 \times 8}{100}$$

$$= 15 \text{ m}^3$$

$$1 \text{ m}^3 = 100$$

$$15 \text{ m}^3 = 1500$$

$$\frac{1000 \text{ bricks}}{1500} = 2000$$

$$\frac{2000 \times 1000}{1000}$$

$$= 3000$$