26/10/18

Maxima and Minima

if $f(x,y)$ be any function of 2 independent variables x, y suppose to be continious for all values in the interval (a, b) then

$f(a,b)$ is said to be maximum or minimum value of $f(x,y)$ according as $f(a+h, b+k)$ is lesser or greater than $f(a,b)$

max $\Rightarrow$ $f(a,b) \geq \lim_{\substack{h \to 0 \\ k \to \infty}} f(a+h, b+k)$

min. $f(a,b) \leq f(a+h, b+k)$

for all sufficently small independent variables h and k provided both h and k are not equal to zero

* Necessary condition for the existance of maxima or minima.

→ Necessary cond^n that the a f^n $f(x,y)$ should have max. or min. value at (a, b) is

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0 \quad , \& \quad \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0$$

Necessary → (get extreme point)
Extreme point value.
give maxima and minima 201

Sufficient condⁿ for maxima or minima:

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} = r \quad , \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)} = s$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} = t$$

Case : 1 if $(rt - s^2) > 0$ \& $r < 0$
then $f(x,y)$ will have maximum value at $x=a$, $y=b$, or (a,b)

Case : 2 if $(rt - s^2) > 0$ \& $r > 0$
then $f(x,y)$ will have minimum valut at (a,b)

Case : 3 if $(rt - s^2) < 0$
the $f(x,y)$ have neither max. nor minima at (x,y)

Case : 4 if $(rt - s^2) = 0$
this case is doubt full and we need further investigation to check wether the fn is max. or minimum value at (a,b)

* Stationary point and Extreame point :

The points which satisfy the condⁿ

$$\left(\frac{\partial f}{\partial x}\right) = 0 \quad \& \quad \left(\frac{\partial f}{\partial y}\right) = 0$$

are called stationary point and the stationary points which will give either maximum or minimum value of the fⁿ that stationary point are called extreames point

Saddle point ⇒ → give neither max. or ~~neither~~ nor min.

Que: find the Extreme value of the fⁿ

$$u = xy(a - x - y)$$

for existance of max. or min.

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = y(a - x - y) + xy(-1) = 0 \quad \&$$

$$\frac{\partial u}{\partial y} = x(a - x - y) + xy(-1) = 0$$

$$ay - 2xy - y^2 = 0 \quad \& \quad ax - x^2 - 2xy = 0$$

$$y(a - 2x - y) = 0 \qquad x(a - x - 2y) = 0$$

$y = 0, \quad x = 0$

$y = 0, \quad \cancel{x=0}\ a - x - 2y = 0$

$a - 2x - y = 0 \ \& \ x = 0$

$a - 2x - y = 0 \ \& \ a - x - 2y = 0$

$(0,0), (a,0) (0,a) \left(\frac{a}{3}, \frac{b}{3}\right)$

$$r = \frac{\partial^2 u}{\partial x^2} = -2y$$

$$t = \frac{\partial^2 u}{\partial y^2} = -2x$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = a - 2x - 2y$$

(i) at $(0,0)$

$r = 0, \quad s = a, \quad t = 0$

$rt - s^2 = -a^2 < 0$

$\Rightarrow$ u has neither max. nor minimum.

(ii) at $(a,0)$

$r = 0, \quad s = -a, \quad t = -2a$

$rt - s^2 = \cancel{a} - a^2 < 0$

u has neither max nor min

(iii) at $(0,a)$

$r = -2a, \quad t = 0, \quad s = -a$

$rt - s^2 = -a^2 < 0$

neither max nor min.

at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$r = -\frac{2a}{3}, \quad s = -\frac{a}{3}, \quad t = -\frac{2a}{3}$$

$$rt - s^2 = \frac{4a^2}{9} - \frac{a}{9} = \frac{a^2}{3} > 0$$

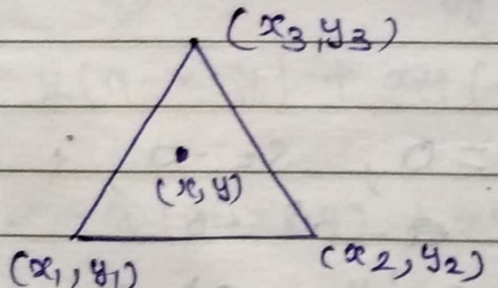
$$r = -\frac{2a}{3} < 0 \text{ if } a > 0$$

u has max. value at $\left(\frac{a}{3}, \frac{a}{3}\right)$

the max value is

$$\frac{a^2}{3 \times 3}\left(a - \frac{2a}{3}\right) = \frac{a^2}{27}$$

Que: find the point with in a triangle such that some of squares of its distances from the three vertices is minimum.



$\sqrt{(x-x_1)^2+(y-y_1)^2}$

$$u = \sum_{r=1}^{3} (x - x_r)^2 + (y - y_r)^2)$$

only 1 st. point not.

$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$

$2(x - x_r)($

$2 \sum_{r=1}^{3} (x - x_r) = 0$

$\sum_{r=1}^{3} 2(y - y_r) = 0$

$2[(x - x_1) + (x - x_2) + (x - x_3)] = 0$

$x = \frac{x_1 + x_2 + x_3}{3}, \quad y = \frac{y_1 + y_2 + y_3}{3}$

(centroid of Δ)

$r = \frac{\partial^2 u}{\partial x^2} = 6$

$t = \frac{\partial^2 u}{\partial y^2} = 6$

$s = \frac{\partial^2 u}{\partial x \partial y} = 0$

$rt - s^2 = 36 - 0 = 0$

$r = 6 > 0$

u has minimum value at

$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$

Que: In a plan triangle find the max. value of

$$y = \cos A \cos B \cos C$$

$$= \cos A \cos B \cos(\pi - (A+B))$$

$$= -\cos A \cos B \cos(A+B)$$

$$\frac{\partial y}{\partial A} = 0 \quad \& \quad \frac{\partial y}{\partial B} = 0$$

$$\frac{\partial y}{\partial A} = -\cos B\left[-\sin A(\cos(A+B)) - \cos A \sin(A+B)\right] = 0$$

$$\frac{\partial y}{\partial B} = -\cos A\left[-\sin B(\cos(A+B)) - \cos B \sin(A+B)\right] = 0$$

$$\cos B \sin(2A+B) = 0 \ \& \ \cos A \sin(A+2B) = 0$$

(i) $\cos B = 0$, $\cos A = 0$ N.V
(ii) $\cos B = 0$ $\cos(A+2B) = 0$ N.V
(iii) $\sin(2A+B) = 0$ $\cos A = 0$ N.V
(iv) $\sin(2A+B) = 0$ $\sin(A+2B) = 0$

$$2A+B = \pi \qquad A+2B = \pi$$

$$A = \frac{\pi}{3}, \quad B = \frac{\pi}{3}$$

$$r = \frac{\partial^2 u}{\partial A^2} = 2\cos B \cos(2A+B)$$

$$s = \frac{\partial^2 u}{\partial A \partial B} = -\sin B \sin(2A+B) + \cos B \cdot \cos(2A+B)$$

$$t = \frac{\partial^2 u}{\partial B^2} = 2\cos A \cos(A+2B)$$

at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$r = -1$

$s = -1/2$

$t = -1$

$$rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

u has max. value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

The max. value is $\frac{1}{8}$

* Lagrange's method of Multiplier (for three variables):

(1) Let $u = f(x_1, x_2, x_3)$ — (1)

u is variable of three ind. variable which are connected by a relation

$$\phi(x_1, x_2, x_3) = 0 \quad - (2)$$

diff. eq^n (1)

we get

$$0 = du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 \quad -(3)$$

diff. eq^n (2)

$$\frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3 = 0 \quad -(4)$$

eq^n (3) + ((4) × λ)

$$\left(\frac{\partial u}{\partial x_1} + \lambda \frac{\partial \phi}{\partial x_1}\right) dx_1 + \left(\frac{\partial u}{\partial x_2} + \lambda \frac{\partial \phi}{\partial x_2}\right) dx_2 + \left(\frac{\partial u}{\partial x_3} + \lambda \frac{\partial \phi}{\partial x_3}\right) dx_3 = 0 \quad (5)$$

eq^n (5) is satisfied only when

$$\left(\frac{\partial u}{\partial x_1} + \lambda \frac{\partial \phi}{\partial x_1}\right) = 0 \ \& \ \left(\frac{\partial u}{\partial x_2} + \lambda \frac{\partial \phi}{\partial x_2}\right) = 0$$

$$\left(\frac{\partial u}{\partial x_3} + \lambda \frac{\partial \phi}{\partial x_3}\right) = 0$$

these three conditions along with eq^n no. (2) will give the stationary point

Que: find the max value of in case of plane Δ

$$u = \cos A \cos B \cos C$$

(1) $\log u = \log \cos A + \log \cos B + \log \cos C$ — (1)

$$A + B + C = \pi \quad \text{— (2)}$$

(2) — diff. eq^n (1)

$$0 = \frac{1}{u} du = \frac{-\sin A}{\cos A} dA - \frac{\sin B}{\cos B} dB - \frac{\sin C}{\cos C} dC$$

$$0 = du = -u\left[\tan A\, dA + \tan B\, dB + \tan C\, dC\right)$$

$$0 = \tan A\, dA + \tan B\, dB + \tan C\, dC. \quad \text{— (3)}$$

(3) — Diff eq^n (2)

$$dA + dB + dC = 0 \quad \text{— (4)}$$

(4) $3 + (4 \times \lambda)$

$$(\tan A + \lambda) dA + (\tan B + \lambda) dB + (\tan C + \lambda) dC = 0$$

— (5)

$\tan A + \lambda = 0$

$\tan B + \lambda = 0$

$\tan C + \lambda = 0$

$\tan A = \tan B = \tan C = -\lambda$

[stationary point] $\Rightarrow A = \frac{\pi}{3} = B = C$

$A+B+C = \pi$

$\underline{C = \pi - A - B}$

Part. diff. eq^n (1) w.r.t A

$$\frac{1}{u}\frac{\partial u}{\partial A} = -\frac{\sin A}{\cos A} - \frac{\sin C}{\cos C}\frac{\partial C}{\partial A}$$

P. diff. (2) w.r.t A

$$1 + \frac{\partial C}{\partial A} = 0$$

$$\boxed{\frac{\partial C}{\partial A} = -1}$$

$$\frac{1}{u}\frac{\partial u}{\partial A} = -\tan A + \tan C$$

P. diff. w.r.t A

$$-\frac{1}{u^2}\left(\frac{\partial u}{\partial A}\right)^2 + \frac{1}{u}\left(\frac{\partial^2 u}{\partial A^2}\right) = -\sec^2 A - \sec^2 C.$$

$$\frac{\partial^2 y}{\partial A^2} = -4(\sec^2 A + \sec^2 C) < 0$$

$\gamma < 0$ so, max.

Que: divide a no. a into three parts such that their product will be max.

$$u = x \cdot y \cdot z$$

$$\log u = \log x + \log y + \log z \quad \text{--- (1)}$$

$$x + y + z = a \quad \text{--- (2)}$$

Diff eq^n (1)

$$\frac{1}{u} du = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$$

$$0 = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \quad \text{--- (3)}$$

diff eq^n (2)

$$dx + dy + dz = 0 \quad \text{--- (4)}$$

$3 + (\lambda \times 4)$

$$\left(\frac{1}{x} + \lambda\right) dx + \left(\frac{1}{y} + \lambda\right) dy + \left(\frac{1}{z} + \lambda\right) dz = 0$$

$$\frac{1}{x} + \lambda = 0, \quad \frac{1}{y} + \lambda = 0, \quad \frac{1}{z} + \lambda = 0$$

$$x = -\frac{1}{\lambda} = y = z$$

$xy = (y+x)dx$

$$x+y+z=0$$

$$-\frac{3}{\lambda}=a$$

$$\lambda = -\frac{3}{a}$$

$$x = \frac{a}{3} = y = z$$

Imp:

Q. P.T All rectangular Parallelopiped of same volume the cube has least surface

$$u = 2xy + 2yz + 2zx \quad \text{—(1)}$$

$$2dxy + 2xdy + 2dyz + 2ydz + 2dzx + 2zdx$$

$$xyz = V = \text{Const}$$

$$\log x + \log y + \log z = \log V \quad \text{—(2)}$$

Diff eq^n (1)

$$2(y+z)dx + (x+z)2dy + (x+y)2dz = 0 \quad \text{—(3)}$$

Diff eq^n (2)

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \quad \text{—(4)}$$

$((3) + ((4) \times \lambda))$

$$\left((y+z) + \frac{\lambda}{x}\right)dx + \left((x+z) + \frac{\lambda}{y}\right)dy + \left((x+y) + \frac{\lambda}{z}\right)dz = 0$$

$$\frac{\partial^2 y}{\partial A^2} = -4(\sec^2 A + \sec^2 C) < 0$$

$r < 0$ so, max.

Que: divide a no. a into three parts such that their product will be max.

$$u = x \cdot y \cdot z$$

$$\log u = \log x + \log y + \log z \quad — (1)$$

$$x + y + z = a \quad — (2)$$

Diff eq^n (1)

$$\frac{1}{u} du = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$$

$$0 = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \quad — (3)$$

diff eq^n (2)

$$dx + dy + dz = 0 \quad — (4)$$

$3 + (\lambda \times 4)$

$$\left(\frac{1}{x} + \lambda\right)dx + \left(\frac{1}{y} + \lambda\right)dy + \left(\frac{1}{z} + \lambda\right)dz = 0$$

$$\frac{1}{x} + \lambda = 0, \quad \frac{1}{y} + \lambda = 0, \quad \frac{1}{z} + \lambda = 0$$

$$x = -\frac{1}{\lambda} = y = z$$

$$x+y+z=0$$

$$-\frac{3}{\lambda}=a$$

$$\lambda=-\frac{3}{a}$$

$$x=\frac{a}{3}=y=z$$

Imp:

Q. P.T All rectangular Parallelopiped of same volume the cube has least surface

$$u=2xy+2yz+2zx \quad -(1)$$

$$2dxy+2xdy+2dyz+2ydz+2dzx+2zdx$$

$$xyz=V=\text{Const} \quad -2$$

$$\log x+\log y+\log z=\log V \quad -(2)$$

Diff eq^n (1)

$$2(y+z)dx+(x+z)dy+(x+y)2dz=0 \quad -(3)$$

Diff eq^n (2)

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0 \quad -(4)$$

$$(3+(4\times\lambda))$$

$$\left((y+z)+\frac{\lambda}{x}\right)dx+\left((x+z)+\frac{\lambda}{y}\right)dy+\left((x+y)+\frac{\lambda}{z}\right)dz=0$$

$$\left((y+z)+\frac{\lambda}{x}\right)=0, \quad \left((x+z)+\frac{\lambda}{y}\right)=0$$

$$\left((x+y)+\frac{\lambda}{z}\right)=0$$

$$\lambda = -x(y+z) = -(xy+xz)$$
$$= -(xy+yz)$$
$$= -(xz+zy)$$

$$\Rightarrow x=y=z=v^{1/3}$$

Rmk:

Que: S.T the volume of greatest rectangular Perallopipe that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is $\frac{8abc}{3\sqrt{3}}$

$$V = xyz$$

$$\log v = \log x + \log y + \log z \quad -(1)$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad -(2)$$

Imp.

Q. Find the max. and min dis. of the point (3, 4, 12) from the square $x^2+y^2+z^2=1$

Two stationary point

$$D = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

$$4 = D^2 = (x-3)^2+(y-4)^2+(z-12)^2$$

only for taken scaller

scaller — vector Gradient

$\hat{a} = \frac{\vec{a}}{|a|}$

* Gradient of scalar func^n :

Let $f(x, y, z)$ be defined differentiable f^n at each point x, y, z in a certain region the gradient of f is denoted by Δf and it defined as

$$\nabla \quad \Delta = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right)$$

$$\Delta f = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) f$$

$$= \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right)$$

Gradient vector is always the normal vector to the surface

* Directional derivative :

The component of $\Delta \phi$ in dir^n of $\vec{a}$ is given by

$$\Delta \phi \cdot \hat{a}$$

and is called dir^n derivative of ϕ in dir^n of $\vec{a}$.

Que. find the dir^n derivative of

$\phi = x^4 + y^4 + z^4$ at point $A(1,-2,1)$ in dir^n of AB where B is $(2,6,-1)$ find the max. dir^n derivative of ϕ at $(1,-2,1)$

$\nabla\phi$)

$$\phi = x^4 + y^4 + z^4$$

$$DD = \Delta\phi \cdot \hat{a}$$

$$= |\Delta\phi| \cos\theta$$

max. value of D.D $= |\Delta\phi|$

$$\nabla\phi = \frac{\partial}{\partial x}(x^4+y^4+z^4)\hat{i} + \frac{\partial}{\partial y}(x^4+y^4+z^4)\hat{j} + \frac{\partial}{\partial z}(x^4+y^4+z^4)\hat{k}$$

$$\nabla\phi = 4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}$$

$$\vec{AB} = 1\hat{i} + 8\hat{j} - 2\hat{k}$$

D.D of ϕ dir^n of $\vec{AB} = \Delta\phi \cdot (\hat{AB})$

$$(4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}) \cdot \left(\frac{\hat{i}+8\hat{j}-2\hat{k}}{\sqrt{69}}\right)$$

$$\frac{4x^3 + 32y^3 - 8z^3}{\sqrt{69}}$$

32×8 $\quad$ 64
246

at point $(1,-2,1)$

$$= \frac{4 + 246 - 8}{\sqrt{69}}$$

iii) max. D.D $= |\Delta \phi|$ at $(1,-2,1)$

$$= |4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}|$$

$$= |4(\hat{i} - 8\hat{j} + \hat{k})|$$

$$= 4(\sqrt{66})$$

Q. if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad r = |\vec{r}|$

P.T $\quad \text{grad}(r) = \frac{\vec{r}}{r}$

$$\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

i) $\nabla r = \sqrt{x^2 + y^2 + z^2}$

$$= \frac{\partial}{\partial x}(r)\hat{i} + \frac{\partial}{\partial y}(r)\hat{j} + \frac{\partial}{\partial z}(r)\hat{k}$$

$\frac{\partial r}{\partial x}$ $\quad = \quad r^2 = x^2 + y^2 + z^2$

$2r\,\partial r = 2x$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} = \frac{\vec{r}}{r}$$

$$\nabla\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)k$$

$$\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)\hat{i} + \left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right)\hat{j} + \left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)\hat{k}$$

$$\left(-\frac{x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k}\right)$$

$$= -\frac{\vec{r}}{r^3}$$

Imp:

Q. Pr. That $\Delta \times [f(r)\,\vec{r}] = 0$

$$f(r)\,\vec{r} = x f(r)\hat{i} + y f(r)\hat{j} + z f(r)\hat{k}$$

$$\Delta = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\begin{matrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x f(r) & y f(r) & z f(r) \end{matrix}$$

$$\hat{i}\left[z\frac{\partial f(r)}{\partial y} - y\frac{\partial f(r)}{\partial z}\right] - \hat{j}\left[z\frac{\partial f(r)}{\partial x} - x\frac{\partial f(r)}{\partial z}\right]$$

$$-\hat{k}\left[y\frac{\partial f(r)}{\partial x} - x\frac{\partial f(r)}{\partial y}\right]$$

$$\frac{\partial f(r)}{\partial y} = f'(r) \times \frac{yz}{r} - f'(r)\frac{yz}{r}$$

$$= 0$$

$$\frac{\partial}{\partial y}\left[f(r)\right] = f'(r) \times \frac{\partial r}{\partial y} = f'(r) \times \frac{y}{r}$$

$$\frac{\partial}{\partial z}(f(r)) = f'(r) \times \frac{\partial r}{\partial z} = f'(r) \times \frac{z}{r}$$

Que If $u = x+y+z$

$v = x^2+y^2+z^2$

$w = xy+yz+zx$

(Scalar triple product $= 0$)

P.T $\nabla u, \nabla v, \nabla w$ are coplanar vectors.

grad u

$$= \frac{\partial}{\partial x}(x+y+z)\hat{i} + \frac{\partial}{\partial y}(x+y+z)\hat{j} + \frac{\partial}{\partial y}(x+y+z)\hat{k}$$

grad $= \hat{i}+\hat{j}+\hat{k}$.

$a.(b \times c) = 0$

$[a\,b\,c] = a.(b \times c) =$

$(a \times b).c = 0$

gradient

Q. Find the unit normal vector to the surface

$$xy^3z^2 = 4 \quad at \quad (-1,-1,2)$$

$$u = xy^3z^2 - 4$$

$$\nabla(u) = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

at $(-1, -1, 2)$

$$-4\hat{i} - 12\hat{j} + 4\hat{k}$$

the unit normal vector to the surface u is

$$= \frac{\nabla u}{|\nabla u|} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16+144+16}}$$

$$= \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{}}$$

Q. Find the angle b/w the surfaces

$$x^2 + y^2 + z^2 = 9$$

and $z = x^2 + y^2 - 3$

at $(2, -1, 2)$

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$u = x^2 + y^2 + z^2 - 9 = 0$

$v = z = x^2 + y^2 - 3$

$\nabla u = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

at $(2, -1, 2)$

$= 4\hat{i} - 2\hat{j} + 4\hat{k}$

$\nabla v = 2x\hat{i} + 2y\hat{j} - 1\hat{k}$

at $(2, -1, 2)$

$\nabla v = 4\hat{i} - 2\hat{j} - \hat{k}$

$\cancel{\Delta u \cdot \Delta v = |\Delta u||}$

$\nabla u \cdot \nabla v = |\nabla u||\nabla v| \cos\theta$

$(16 + 4 - 4) = \sqrt{16+4+16}\sqrt{16+4+1} \cos\theta$

$\frac{16}{\sqrt{36}\sqrt{21}} = \cos\theta = \frac{16}{6\sqrt{21}}$

$\theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right) = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

Point

* Divergence of vector function :

$$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$$

$$div(\vec{F}) = \nabla . \vec{F}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$= \cancel{\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}} +$$

$$= \frac{\partial}{\partial x}f_1 + \frac{\partial}{\partial y}f_2 + \frac{\partial}{\partial z}F_3$$

Imp

Scalar f^n.

* If $div(\vec{F})$ is zero then $\vec{F}$ is called solenoidal vector.

Imp

Que: Find the value of n for which the vector $r^n \vec{r}$ is solenoidal

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$div(r^n \vec{r}) = 0$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla . (r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k}) = 0$$

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$$\frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) = 0$$

$$\frac{\partial}{\partial x}(r^n x) = r^n + n x r^{n-1} \times \frac{\partial r}{\partial x} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= r^n + n x^2 r^{n-2}$$

$$r^n + n x^2 r^{n-2} + r^n + n y^2 r^{n-2} + r^n + n z^2 r^{n-2}$$

$$3r^n + n r^{n-2}(x^2 + y^2 + z^2)$$

$$3r^n + n r^n = 0$$

$$(n+3) r^n = 0$$

$$\boxed{n = -3}$$

Que: Find the D.D of the divergence of

$$f = xy\hat{i} + 2xy^2\hat{j} + z^2\hat{k}$$

at $(2, 1, 2)$ in the dirn of normal to the sphere $(x^2 + y^2 + z^2 = 9)$.

$$\phi = \text{div } F = (y + 2xy + 2z)$$

$$\text{grad } \phi = 2y\hat{i} + (1 + 2x)\hat{j} + 2k$$

at $(2, 1, 2)$

$$\text{grad } \phi = 2\hat{i} + 5\hat{j} + 2k$$

$\vec{a} = \nabla(x^2+y^2+z^2-9)$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

at $(2,1,2)$

$\vec{a} = 4\hat{i} + 2\hat{j} + 4\hat{k}$

DD of div f in dirn of $\vec{a}$

$$\vec{a} = \frac{(2\hat{i} + 5\hat{j} + 2\hat{k}) \cdot (4\hat{i} + 2\hat{j} + 4\hat{k})}{\sqrt{16+4+16}}$$

$$= \frac{8+10+8}{6} = \frac{26}{6}$$

Q. P.T $\quad \text{div}(\text{grad}\, r^n) = n(n+1)r^{n-2}$

$r = \sqrt{x^2+y^2+z^2}$

also show that $\nabla\left(\frac{1}{r}\right) = 0$

$$\text{grad}(r^n) = nr^{n-1}\frac{x}{r}\hat{i} + nr^{n-1}\frac{y}{r}\hat{j} + nr^{n-1}\frac{z}{r}\hat{k}$$

$$\text{grad}(r^n) = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$\text{div.}\left[\text{grad}\,(r^n)\right] =$

$= \frac{\partial}{\partial x}\left(n r^{n-2} x\right) + \frac{\partial}{\partial y}\left(n r^{n-2} y\right) + \frac{\partial}{\partial z}\left(n r^{n-2} z\right)$

$\frac{\partial}{\partial x}\left(n r^{n-2} x\right) = n\left(r^{n-2} + x(n-2) r^{n-3} \frac{x}{r}\right)$

$\text{div}(\text{grad}\,(r^n)) = n\left[3r^{n-2} + (n-2) r^{n-4} (x^2+y^2+z^2)\right]$

$= n\left[3r^{n-2} + (n-2) r^{n-2}\right.$

$= n r^{n-2}\left[3+n-2\right]$

$= n(n+1) r^{n-2}$

$\nabla^2 \cdot \frac{1}{r} = \nabla \cdot \left(\nabla \cdot \left(\frac{1}{r}\right)\right)$

$= 0$

$\left(= \left(\text{div}\left(\text{grad}\left(\frac{1}{r}\right)\right)\right)\right)$

* Curl of a vector point fn:

$$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3 k$$

$$\text{curl}(\vec{F}) = \nabla \times \vec{f}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

if $\text{curl}(\vec{f}) = 0 \Rightarrow \boxed{\nabla \times \vec{f} = 0}$

$\vec{F}$ is called irrotational vector

Imp.
Que:

P.T $r^n \vec{r}$ is irrotational vector

$$\text{curl}(r^n \vec{r}) = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{d}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \hat{i}\left[\left(\frac{\partial}{\partial y}(r^n z)\right) - \frac{\partial}{\partial z}(r^n y)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right]$$

$$= \hat{i}\left[z n r^{n-2} y - n y r^{n-2} z\right]$$

$$= 0$$

<u>Que:</u> if a vector field is given by

$$\vec{f} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$$

is this field irrotational if so, find the <u>scalar potential</u> (ϕ)

$$\text{curl } \vec{F} = 0$$

$$\vec{f} = \nabla \phi$$

$$\vec{F} = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

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Rolle's theorem

Review
Roll
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$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$d\phi = f \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$d\phi = (x^2 - y^2 + x)dx - (2xy + y)dy$$

$$\phi = \frac{x^3}{3} - xy^2 + \frac{x^2}{2} - xy^2 - \frac{y^2}{2}$$

(1) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$

(2) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log|x + \sqrt{x^2-a^2}| + C$

(3) $\int \frac{1}{\sqrt{a^2+x^2}}dx = \log|x + \sqrt{a^2+x^2}| + C$

(4) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(5) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x + \sqrt{x^2-a^2}|$

(6) $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log|x + \sqrt{a^2+x^2}|$

(7) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(8) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(9) $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$