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Exponential Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Put $x = 1$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$= 1 + 1 + 0.5 + \cdots$$

$$= 2.7182$$

* $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^1$

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots$$

$$= \left[1 + {}^nC_1\left(\frac{1}{n}\right) + {}^nC_2 \frac{1}{n^2} + \cdots\right]$$

$$= \left[1 + 1 + \frac{n(n-1)}{2!\,n^2} + \frac{n(n-1)(n-2)}{3!\,n^3} \cdots\right]$$

$$\left(1 + \frac{1}{n}\right)^n = 1 + 1 + \frac{(1 - 1/n)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} \cdots)$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right]$$

$$= e$$

* Proof that

$$a^x = 1 + x\log a + \frac{x^2(\log a)^2}{2!} + \frac{x^3(\log a)^3}{3!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x\log a} = \left[1 + (x\log a) + \frac{x^2(\log a)^2}{2!} \cdots\right]$$

$$e^{\log a^x} = \left[1 + (x\log a) + \frac{x^2(\log a)^2}{2!} \cdots\right]$$

$$a^x = \left[1 + (x\log a) + \frac{x^3(\log a)^2}{2!} \cdots\right]$$

* <u>Results</u>

(1) $e^x = \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$

(2) $e^{-x} = \left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]$

(3) $e^x + e^{-x} = 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right]$

$$\frac{e^x + e^{-x}}{2} = \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right]$$

(3) $\frac{e^{x}+e^{-x}}{2}=\left[1+\frac{x^2}{2!}+\frac{x^4}{4!}+---\right]$

(4) $\frac{e^{x}-e^{-x}}{2}=\left[\frac{x^{\circ}}{1!}+\frac{x^3}{3!}+---\right]$

(5) $\frac{e+e^{-1}}{2}=\left[1+\frac{1}{2!}+\frac{1}{4!}+---\right]$

(6) $\frac{e-e^{-1}}{2}=\left[\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+---\right]$

*Q. Write the expension of

$\frac{e^{3x}-e^{-3x}}{2}=\left[\frac{1}{1!}+\frac{3x}{1!}\right.$

$e^{3x}=\left[1+\frac{3x}{1!}+\frac{3^2x^2}{2!}+---\right]$

$e^{-3n}=\left[1-\frac{3n}{1!}+\frac{3^2n^2}{2!}+---\right]$

$e^{3n}-e^{-3n}=2\left[\frac{3n}{1!}+\frac{3^2n^2}{3!}+--\right]$

* $\frac{e^{5n}+e^{n}}{e^{2n}}=$

Q. Show that

$$\frac{e-1}{e+1} = \frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+ \cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+ \cdots}$$

$$\left[\frac{e+e^{-1}}{2}-1\right] = \left[1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+ \cdots\right]-1$$

$$\frac{e^2+1-2e}{2e} \quad \text{—— (1)}$$

$$\left[\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+ \cdots\right]$$

$$= \frac{e-e^{-1}}{2} =$$

$$\frac{e^2-1}{2e}$$

$$= \frac{e^2+1-2e}{e^2-1}$$

$$= \frac{e-1}{e+1}$$

$$\boxed{-r! = \infty}$$

***Proof that**

$$1 + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \frac{1+2+3+4}{4!} + \cdots = \frac{3e}{2}$$

$$t_n = \frac{1+2+3+4+\cdots n}{n!} = \frac{n(n+1)}{2n!}$$

Remove n from the Numerator

$$= \frac{n+1}{2(n-1)!} = \frac{n-1+2}{2(n-1)!}$$

$$= \frac{1}{2(n-2)!} + \frac{1}{(n-1)!}$$

$$= \frac{1}{2}\left[\frac{1}{(n-2)!} + \frac{2}{(n-1)!}\right]$$

$$t_1 = \frac{1}{2}\left[\frac{1}{-1!} + \frac{2}{0!}\right] = \frac{1}{2}\left[\frac{1}{\infty} + \frac{2}{0!}\right] = \frac{1}{2}[0+2]$$

$$t_2 = \frac{1}{2}\left[1 + \frac{2}{1!}\right]$$

$$t_3 = \frac{1}{2}\left[\frac{1}{1!} + \frac{2}{2!}\right]$$

$$t_1+t_2+t_3 = \frac{1}{2}\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$= \frac{e}{2} + e = \frac{3e}{2}$$

Que: $9 + \frac{16}{2!} + \frac{27}{3!} + \text{------}$

$S = 9 + 16 + 27 + \text{-----} \; t_n$

$S = \quad 9 + 16 + 27 + \text{---} \; t_{n-1} + t_n$

$t_n = 9 + [7 + 11 + \text{---} \; t_{n-1}]$

$= 9 + [\frac{n-1}{2}(14 + (n-2)4)]$

$= 2n^2 + n + 6$

$T_n = \frac{2n^2 + n + 6}{n!} = \frac{2(n)(n-1) + 3n + 6}{n!}$

$= \frac{2}{(n-2)!} + \frac{3}{(n-1)!} + \frac{6}{n!}$

$T_1 = 0 + 3 + \frac{6}{1!}$

$\cancel{T_2} =$

$T_2 = 2 + \frac{3}{1!} + \frac{6}{2!}$

$T_3 = \frac{2}{1!} + \frac{3}{2!} + \frac{6}{3!}$

$= 2\left[1 + \frac{1}{1!} + \text{--}\right] + 3\left[1 + \frac{1}{1!} + \frac{1}{2!} + \text{--}\right]$

$+ 6\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \text{---}\right]$

$= 2e + 3e + 6e - 6$

$= 11e - 6$

$\left({}^nC_r = \frac{n!}{r!(n-r)!} \right.$

$\left. {}^nP_r = \frac{n!}{(n-r)!} \right)$

Que:

$$\sum_{n=2}^{\infty} \frac{{}^nC_2}{(n+1)!}$$

Find this value of this Series

Remove n from

$$\frac{n(n-1)}{2!(n+1)!}$$

$$\frac{n((n+1)-2)}{2!(n+1)!}$$

$$\frac{1}{2!}\left[\frac{n(n+1)}{(n+1)!} - \frac{2n}{(n+1)!}\right]$$

$$\frac{1}{2!}\left[\frac{1}{(n-1)!} - \frac{2((n+1)-1)}{(n+1)!}\right]$$

$$\frac{1}{2!}\left[\frac{1}{(n-1)!} - \frac{2}{n!} + \frac{2}{(n+1)!}\right]$$

$$T_1 = \frac{1}{2!}\left[\frac{1}{1!} - \frac{2}{2!} + \frac{2}{3!}\right]$$

$$T_2 = \frac{1}{2!}\left[\frac{1}{2!} - \frac{2}{3!} + \frac{2}{4!}\right]$$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots {}^nC_n x^n]$

$T_1 + T_2 + \cdots$

$\frac{1}{2!}\left[\left(\frac{1}{1!} + \frac{1}{2!} + \cdots\right] - 2\left(\frac{1}{2!} + \frac{1}{3!} + \cdots\right)\right.$

$\left. + 2\left(\frac{1}{3!} + \frac{1}{4!} + \cdots\right)\right]$

$= \frac{1}{2}\left[(e-1) - 2(e-2) + 2\left(e-2-\frac{1}{2}\right)\right]$

$= \frac{1}{2}\left[e - 1 - \cancel{2e} + \cancel{4} + \cancel{2e} - \cancel{4} - 1\right]$

$= \frac{1}{2}[e-2]$

Q. $\sum_{n=1}^{\infty} \frac{{}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots {}^nC_n}{{}^nP_n}$

$(1+x)^n = [{}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots {}^nC_n x^n)$

$2^n = ({}^nC_0 + {}^nC_1 + \cdots {}^nC_n)$

$t_n = \frac{2^n}{n!}$

$\left[\frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots\right] - 1$

$$\left[1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\cdots\right]-1$$

$$e^x = 1+x+\frac{x^2}{2!}+\cdots$$

$$(e^2-1)$$

* Logarithm Series:

1) $\log(1+x) = \left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\cdots\right]$

Replace $x=-x$

2) $\log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right]$

3) $\frac{\log(1+x)}{\log(1-x)} = \log(1+x)-\log(1-x)$

$$\frac{\log(1+x)}{\log(1-x)} = \left[2x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right]$$

4) Put $x=\frac{1}{n}$

$$\frac{\log(n+1)}{\log(n-1)} = 2\left[\frac{1}{n}+\frac{1}{n^3 3}+\frac{1}{n^5 5}+\cdots\right]$$

Q. Proff that $\left[\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{3}\cdot\frac{1}{2^3} - \right.$

$\log(1+x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots\right]$

$\log\left(1+\frac{1}{2}\right) = \log\left(\frac{3}{2}\right)$

Q. Proff that $\left[\frac{1}{4} - \frac{1}{2}\frac{1}{4^2} + \frac{1}{3}\frac{1}{4^3} + - - \right]$

$= \frac{1}{5} + \frac{1}{2}\frac{1}{5^2} + \frac{1}{3}\cdot\frac{1}{5^2} + - \Big]$

Sol^n:

$\log\left(1+\frac{1}{4}\right) = \log\left(\frac{5}{4}\right) \quad - (1)$

$-\log(\cancel{0}1-x) = -\log\left(1-\frac{1}{5}\right)$

$= -\log\left(\frac{4}{5}\right)$

$= \log\left(\frac{5}{4}\right) - (2)$

Q. $\frac{a-b}{a} + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 \ldots = \log\left(\frac{a}{b}\right)$

$= -\log\left(1-\left(\frac{a-b}{a}\right)\right)$

$= -\log\left(\frac{b}{a}\right)$

$= \log\left(\frac{a}{b}\right)$

Q. $1 + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \frac{1}{5}\left(\frac{1}{2}\right)^4 + \ldots = \log 3$

$= 2\frac{1}{2}\left[1 + \frac{1}{3}\left(\frac{1}{2}\right)^2 + \ldots\right]$

$= 2\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \ldots\right]$

$= \log\left(\frac{\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}\right)$

$= \log 3.$

Q. $\log(1+3x+2x^2) =$

$= 3x - \frac{5}{2}x^2 + 3x^2 - \frac{17}{4}x^4 + \ldots$

$= \log((2x+1)(x+1))$

$= \log(2x+1) + \log(x+1)$

$= \left[2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \cdots\right] +$

$+ \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]$

$= 3x - \frac{5x^2}{2} + 3x^2$ H.P.d

Que: If $y = 2x^2 - 1$

$\frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} \cdots = \frac{2}{y} + \frac{2}{3y^2} + \frac{2}{5y^5} +$

RHS $= 2\left[\frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \cdots\right]$

$= \log\left(\frac{1+1/y}{1-1/y}\right) = \log\left(\frac{1-1/y}{1+1/y}\right)$

$= \log\left(\frac{y+1}{y-1}\right)$

$\log\left(\frac{2x^2}{2x^2-2}\right) = \log\left(\frac{x^2}{x^2-1}\right)$

$$= -\log\left(\frac{x^2-1}{x^2}\right)$$

$$= -\log\left(1-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2} + \frac{1}{x^4 2} + \frac{1}{x^6 3} + \cdots$$

Que: $2\log n - \log(n+1) - \log(n-1)$

$$= 2\left[\frac{1}{2n^2-1} + \frac{1}{3(2n^2-1)^3} + \frac{1}{5(2n^2-1)^5} + \right.$$

Let $\frac{1}{(2n^2-1)} = y$

$$= \log\left(\frac{1+y}{1-y}\right)$$

$$= \log\left(\frac{1+ 1/2n^2-1}{1- 1/2n^2-1}\right)$$

$$= \log\left(\frac{2n^2}{2n^2-2}\right)$$

$$= \log\left(\frac{n^2}{n^2-1}\right)$$

$$= \log\left(\frac{n^2}{(n+1)(n+1)}\right)$$

$$\log(n^2) - \log(n+1) - \log(n+1)$$

$$= 2\log n - \log(n+1) - \log(n+1).$$

Q. $\log 10 = 3\log 2 + \frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 - \frac{1}{3}\left(\frac{1}{4}\right)^3 -$

$$= \log 2^3 + \left[\log\left(1+\frac{1}{4}\right)\right)$$

$$= \log 2^3 + \log\left(\frac{5}{4}\right)$$

$$= \log 8 + \log 5 - \log 4$$

$$= \log\left(\frac{8\cdot 5}{4}\right) = \log 10 \text{ Ans}$$

Que: $\log\left(1+\frac{1}{n}\right)^n = \left(1 - \frac{1}{2(n+1)} - \frac{1}{2\cdot 3(n+1)^2} - \frac{1}{3\cdot 4(n+1)^3} - - \right]$

let $x = \frac{1}{n+1}$

$$= \left[1 - \frac{x}{2} - \frac{x^2}{2\cdot 3} - \frac{x^3}{3\cdot 4} - - \right]$$

=

$$= \left[1-\left(1-\frac{1}{2}\right)x-\left(\frac{1}{2}-\frac{1}{3}\right)x^2-\left(\frac{1}{3}-\frac{1}{4}\right)x^3 \cdots\right]$$

$$-\left[\cancel{0}+x+\frac{x^2}{2}+\frac{x^3}{3}\right]+\left[1+\frac{x}{2}+\frac{x^2}{3}+\cdots\right]$$

$$= -x\left[1+\frac{x}{2}+\frac{x^2}{3}+\cdots\right]+\left[1+\frac{x}{2}+\frac{x^2}{3}+\cdots\right]$$

$$= \left[1+\frac{x}{2}+\frac{x^2}{3}+\cdots\right]\left[1-x\right]$$

$$\left[\frac{1}{x}-\frac{1}{\cancel{x}}\right]x$$

$$= \left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right]\left[\frac{1}{x}-1\right)$$

$$(n+\cancel{1}-\cancel{1})\left(\frac{\cancel{n}}{\cancel{n+1}}\right)\left(-\log\left(1-\frac{1}{n+1}\right)\right)$$

$$= \frac{n}{\cancel{n+1}}\left(-\log\left(\frac{n}{n+1}\right)\right)$$

$$= -n\log\left(\frac{n}{n+1}\right)$$

→ Partial, differentiation.

$$= \log\left(\frac{n}{n+1}\right)^{n}$$